Section 6.1: The Natural Logarithm Function

Question: We know that:

\[
D_x \left( \frac{1}{2} x^2 \right) = x \quad \text{[1]}
\]

\[
D_x (x) = 1 = x^{0} \quad \text{[0]}
\]

\[
D_x (-\frac{1}{x}) = \frac{1}{x^2} = x^{-2} \quad \text{[2]}
\]

But whose derivative is \( x = x^{-1} \)?

**Definition**

The natural logarithm function is defined by

\[
\ln x = \int_{1}^{x} \frac{1}{t} \, dt
\]

Geometric Interpretation:
Derivative

It follows directly from the definition of the log function:

\[(\ln x)' = \frac{1}{x}\]

Properties:

If \(a\) and \(b\) are positive numbers and \(r\) is any rational number, then:

(1) \(\ln 1 = 0\)

(2) \(\ln ab = \ln a + \ln b\)

(3) \(\ln \frac{a}{b} = \ln a - \ln b\)

(4) \(\ln a^r = r \ln a\)

Proof:
Example 1

\[ D_x(\ln (3\sin x - \sqrt{2x + 7})) \]

Example 2

Show that: \[ D_x \ln |x| = \frac{1}{x} \]

Example 3

\[ D_x \left( \frac{1 + \sin^2 x}{(x + 2)^{\frac{3}{2}}} \right) \]

Example 4

Calculate:

(i) \[ \int \tan x \, dx \]

(ii) \[ \int_0^{\frac{\pi}{4}} \tan x \, dx \]
Example 6 Calculate:

\[ \int \frac{t + 1}{2t^2 + 4t + 3} \, dt \]

Section 6.2: Inverse Functions and Their Derivatives

Notation: We use \( y = f^{-1}(x) \) to denote the inverse of \( y = f(x) \)
Inverse Functions

If $f(x)$ and $f^{-1}(x)$ are inverse functions:

- $f(x)$ must be one-to-one, i.e., the inverse exists when we can get back to an $x$ given a $y$. The horizontal line test may be used.
- If $(a, b)$ is on $f(x)$, then $(b, a)$ is on $f^{-1}(x)$.
- $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.
- The domain of $f(x)$ becomes the range of $f^{-1}(x)$.
- The range of $f(x)$ becomes the domain of $f(x)$.

**Theorem**

If $f$ is strictly monotonic on its domain, then $f$ has an inverse.
Example 1

Show that $g(x) = \int_1^x (\cos^2 t + 4t) \, dt$ has an inverse on $(-\infty, +\infty)$, whereas $h(x) = x^2 - 3$ doesn’t.

Example 2

Find the inverse of $f(x) = \frac{x^3 + 2}{x^3 + 1}$ and show that $f(f^{-1}(x)) = x$.

Review: Three steps to find the inverse

- Step 1:
- Step 2:
- Step 3:
Here comes the main theorem of this lecture.

**Theorem**

let $f$ be differentiable and strictly monotonic on some certain domain. If $f'(x) \neq 0$ at a certain $x$ in $I$, then $f^{-1}$ is also differentiable at the corresponding point $y = f(x)$ in the range of $f$ and

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

Which is often written symbolically as

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Interpretation (A way to memorize the theorem):

**Example 3**

Let $f(x) = 3x^5 + x - 2$, find $f^{-1}(2)$. 
Section 6.3: The natural Exponential Function

Definition: The inverse of \( \ln x \) is called the natural exponential function, denoted by \( \exp x \).

Thus, we have the following relation:
\[ x = \exp y \iff y = \ln x \]

Since \( y = \ln x \) and \( y = \exp x \) are inverses of each other, it can be observed immediately that:

- \( \ln(\exp x) = x \)
- \( \exp(\ln x) = x \)

Definition: The letter \( e \) denotes the unique number such that \( \ln e = 1 \).

Fact: \( e \approx \)

Key Observation:
\( e^r = \exp(\ln e^r) = \exp(r \ln e) = expr \)

From the above analysis, we know that \( \ln x \) and \( e^x \) are inverse functions of each other, we have:

1. \( e^{\ln x} = x \)
2. \( \ln(e^x) = x \)

Properties

1. \( e^a e^b = e^{a+b} \)
2. \( \frac{e^a}{e^b} = e^{a-b} \)
3. \( \frac{d}{dx} e^x = e^x \)
Examples

(1) Simplify $e^{\ln v^{2014} - \ln v^{801}}$.

(2) Find $D_x(e^x \cos(\sqrt{x}))$.

(3) Find $\frac{dy}{dx}$ if $2y = e^{xy} - 2\tan x$.

(4) Evaluate the following integrals

(i) $\int_0^1 x^4 e^{-7x^5 + 1} \, dx$

(ii) $\int \frac{2e^x}{e^x - 1} \, dx$

(5) Let $f(x) = \frac{\ln x}{1 + (\ln x)^2}$ for $x \in (0, \infty)$, find:

(a) $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to \infty} f(x)$

(b) the maximum and minimum values of $f(x)$
Section 6.4: General Exponential and Logarithmic Functions

1. Observations: $a^x = \quad$

2. Definition: For $a > 0$ and any real number $x$.
   $$a^x = e^{x \ln a}$$

3. Definition of the inverse function:
   $$y = \log_{a} x \iff x = a^y$$

4. Properties of Exponents:
   If $a > 0$, $b > 0$, and $x$ and $y$ are real numbers,
   (i) $a^x a^y = a^{x+y}$
   (ii) $\frac{a^x}{a^y} = a^{x-y}$
   (iii) $(a^x)^y = a^{xy}$
   (iv) $(ab)^x = a^x b^x$
   (v) $(\frac{a}{b})^x = \frac{a^x}{b^x}$

5. Derivatives and Integrals
   (1) $D_x a^x = a^x \ln a$
   (2) $\int a^x dx = \left( \frac{1}{\ln a} \right) a^x + C$
   (3) $D_x \log_{a} x = \frac{1}{x \ln a}$
Examples

(1) If \( y = x^x \), find \( \frac{dy}{dx} \)

(2) Differentiate:

\( (x^2 + 3)\sin \sqrt{2x} + (\sin \sqrt{2x})^2 + 3 \)

(3) \( \int 3^{4x-2} \, dx \) and \( \int (4x - 2)^3 \, dx \)

(4) \( \int \frac{2\sqrt{x}}{\sqrt{x}} \, dx \)
Section 6.5: Exponential Growth and Decay

Motivation: We consider a population model, which is the simplest one.

**Assumptions**

(1) Population as a function of time \( t \) is differentiable.

(2) The increase \( dy \) in population is proportional to the short time increment \( dt \) and the size of the population at that instant, which is \( y(t) \). \( (dy = k \cdot dt \cdot y(t)) \)

The associate differential equation is:

\[
\frac{dy}{dt} = ky(t)
\]

We now solve this ordinary differential equation.

We begin by rewriting the ordinary differential equation as:

\[
\frac{dy}{y} = kdt
\]

![Graph of exponential growth and decay](image-url)
Example 1. A bacterial population grows at a rate that is proportional to its size. Initially, it is 10,000 and after ten days it is 20,000. What is the population after 25 days?

Example 2 (Theorem). Calculate

$$\lim_{x\to 0} \frac{1}{(1 + x)^x}$$

Example 3. Calculate the following integrals and compare results with Example 2.

(a) 

$$\lim_{x\to 0} (1 + x)^{100000^x}$$

(b) 

$$\lim_{x\to 0} \frac{1}{x}$$
Example 4. Using results of example 2, calculate the following.

(a) \[ \lim_{x \to 0} \frac{1}{(1 + 5x)x} \]

(b) \[ \lim_{n \to \infty} (\frac{n + 2}{n})^n \]
Section 6.8: The Inverse Trigonometric Functions and Their Derivatives

1. Motivation: Recall that for a function to have an inverse, the function has to be one-to-one, that is, it has to pass the horizontal line test on some certain domain. Especially, we know that if a function is monotonic on some domain, then it has an inverse.

Part I: Basic Relations

2. Definition: Inverse Sine and Inverse Cosine (Review)

\[
\begin{align*}
&x = \sin^{-1} y \iff y = \sin x, x \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \\
&x = \cos^{-1} y \iff y = \cos x, x \in [0, \pi] \\
&x = \tan^{-1} y \iff y = \tan x, x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \\
&x = \sec^{-1} y \iff y = \sec x, x \in [0, \pi], x \neq 0
\end{align*}
\]

3. Reciprocal Relations:

\[
\begin{align*}
&\sec^{-1} y = \cos^{-1}\left( \frac{1}{y} \right) \\
&\cot^{-1} y = \tan^{-1}\left( \frac{1}{y} \right) \\
&\csc^{-1} y = \sin^{-1}\left( \frac{1}{y} \right)
\end{align*}
\]

4. Some useful identities:

\[
\begin{align*}
&\sin(\cos^{-1} x) = \sqrt{1 - x^2} \\
&\cos(\sin^{-1} x) = \sqrt{1 - x^2} \\
&\sec(\tan^{-1} x) = \sqrt{1 + x^2} \\
&\tan(\sec^{-1} x) = \text{sgn}(x) \ast \sqrt{x^2 - 1}, \text{ for } |x| \geq 1
\end{align*}
\]

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Examples:

(a) \( \sin^{-1}(\sin \frac{3\pi}{2}) = \)

(b) \( \sin^{-1}(\sin(\frac{\pi}{4})) = \)

(c) \( \sec^{-1}(-1) = \)

(d) \( \tan(2 \tan^{-1}(\frac{1}{3})) = \)

Part II: Derivatives

1. Derivatives of the basic functions: (Review)
   - \( D_x(\sin x) = \)
   - \( D_x(\cos x) = \)
   - \( D_x(\tan x) = \)
   - \( D_x(\sec x) = \)
   - \( D_x(\csc x) = \)
   - \( D_x(\cot x) = \)

2. Derivatives of the inverse functions (New)
   - \( D_x(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \ x \in (-1, 1) \)
   - \( D_x(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \ x \in (-1, 1) \)
   - \( D_x(\tan^{-1} x) = \frac{1}{1+x^2} \)
   - \( D_x(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \ |x| > 1 \)
Examples:

(1) $D_x(x\sin^{-1}(x^3 + 2))$

(2) $D_x(e^{\sqrt{x}\tan^{-1}(x + 2\sin x)})$

Part III: Integrals

New relations:

(1) $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

(2) $\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(3) $\int \frac{1}{x\sqrt{x^2 - a^2}} \, dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

Examples:

(a) $\int_{\sqrt{2}}^{2} \frac{dx}{x\sqrt{x^2 - 1}}$

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Part I: Basic Relations

1. Definition:

\[ \sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2} \]

\[ \tanh x = \frac{\sinh x}{\cosh x} \quad \coth x = \frac{\cosh x}{\sinh x} \]

\[ \text{sech } x = \frac{1}{\cosh x} \quad \csch x = \frac{1}{\sinh x} \]
Example 1: Show that \( \cosh^2 x - \sinh^2 x = 1 \)

**Part II: Derivatives**

(i) \( D_x \sinh x = \)

(ii) \( D_x \cosh x = \)

(iii) \( D_x \tanh x = \)

(iv) \( D_x \coth x = -\text{csch}^2 x \)

(v) \( D_x \text{sech} x = -\text{sech} x \tanh x \)

(vi) \( D_x \text{csch} x = -\text{csch} x \coth x \)

**Examples:**

(1) \[ D_x \cosh^3 (\ln(\sqrt{3x + 2})) \]

(2) \[ \int \tanh x \, dx \]
Part III: Inverse Hyperbolic Functions

1. Basic Relations (You need to know how to derive these relations):

(1) \( \text{sinh}^{-1} x = \ln(x + \sqrt{x^2 + 1}) \)

(2) \( \text{cosh}^{-1} x = \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1 \)

(3) \( \text{tanh}^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x}, \quad -1 < x < 1 \)

(4) \( \text{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right), \quad 0 < x \leq 1 \)

2. Derivatives of the inverse hyperbolic functions:

(1) \( D_x \text{sinh}^{-1} x = \frac{1}{\sqrt{x^2 + 1}} \)

(2) \( D_x \text{cosh}^{-1} x = \frac{1}{\sqrt{x^2 - 1}}, \quad x > 1 \)

(3) \( D_x \text{tanh}^{-1} x = \frac{1}{1 - x^2}, \quad -1 < x < 1 \)

(4) \( D_x \text{sech}^{-1} x = -\frac{1}{x\sqrt{1 - x^2}}, \quad 0 < x < 1 \)
Example: Show that $D_x \sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}}$ by two different methods.