If the population variances are known to be $\sigma_1^2$ and $\sigma_2^2$, then the two-sided confidence interval for the difference of the population means $\mu_1 - \mu_2$ with confidence level $1 - \alpha$ is

\[
\left( \bar{X} - \bar{Y} \right) - z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}, \quad \left( \bar{X} - \bar{Y} \right) + z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}
\]
Samples from Normal Populations with Known Variances

In case of known population variances, the procedures for hypothesis testing for the difference of the population means \( \mu_1 - \mu_2 \) is similar to the one sample test for the population mean:

Null hypothesis \( H_0 : \mu_1 - \mu_2 = \Delta_0 \)

Test statistic value

\[
    z = \frac{(\bar{X} - \bar{Y}) - \Delta_0}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}
\]

Alternative Hypothesis Rejection Region for Level \( \alpha \) Test

\( H_a : \mu_1 - \mu_2 > \Delta_0 \) \hspace{1cm} z \geq z_\alpha \hspace{0.5cm} \text{(upper-tailed)}

\( H_a : \mu_1 - \mu_2 < \Delta_0 \) \hspace{1cm} z \leq -z_\alpha \hspace{0.5cm} \text{(lower-tailed)}

\( H_a : \mu_1 - \mu_2 \neq \Delta_0 \) \hspace{1cm} z \geq z_{\alpha/2} \text{ or } z \leq -z_{\alpha/2} \hspace{0.5cm} \text{(two-tailed)}
Large Size Samples

When the sample size is large, both $\bar{X}$ and $\bar{Y}$ are approximately normally distributed, and

$$Z = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}}$$

is approximately a standard normal rv.
Large Size Samples

In case both \( m \) and \( n \) are large (\( m, n > 30 \)), the procedure for constructing confidence interval and testing hypotheses for the difference of two population means are similar to the one sample case.

The two-sided confidence interval for the difference of the population means \( \mu_1 - \mu_2 \) with confidence level \( 1 - \alpha \) is

\[
\left( (\bar{X} - \bar{Y}) - z_{\alpha/2} \sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}} \right), \quad (\bar{X} - \bar{Y}) + z_{\alpha/2} \sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}
\]
Large Size Samples

In case both $m$ and $n$ are large ($m, n > 30$), the procedures for hypothesis testing for the difference of the population means $\mu_1 - \mu_2$ is:

Null hypothesis $H_0 : \mu_1 - \mu_2 = \Delta_0$

Test statistic value

$$z = \frac{(\bar{X} - \bar{Y}) - \Delta_0}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}}$$

Alternative Hypothesis | Rejection Region for Level $\alpha$ Test
---|---
$H_a : \mu_1 - \mu_2 > \Delta_0$ | $z \geq z_\alpha$ (upper-tailed)
$H_a : \mu_1 - \mu_2 < \Delta_0$ | $z \leq -z_\alpha$ (lower-tailed)
$H_a : \mu_1 - \mu_2 \neq \Delta_0$ | $z \geq z_{\alpha/2}$ or $z \leq -z_{\alpha/2}$ (two-tailed)
Two-Sample \( t \) Test and C.I.

**Theorem**

*When the population distributions are both normal, the standardized variable*

\[
T = \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{S^2_X}{m} + \frac{S^2_Y}{n}}}
\]

*has approximately a \( t \) distribution with df \( \nu \) estimated from the data by*

\[
\nu = \frac{\left(\frac{s^2_X}{m} + \frac{s^2_Y}{n}\right)^2}{\frac{(s^2_X/m)^2}{m-1} + \frac{(s^2_Y/n)^2}{n-1}}
\]

*(round \( \nu \) down to the nearest integer.)*
The **two-sample t confidence interval** for $\mu_X - \mu_Y$ with confidence level $100(1 - \alpha)\%$ is given by

$$
\left( \bar{x} - \bar{y} - t_{\alpha/2, \nu} \sqrt{\frac{s^2_x}{m} + \frac{s^2_y}{n}}, \ (\bar{x} - \bar{y}) + t_{\alpha/2, \nu} \sqrt{\frac{s^2_x}{m} + \frac{s^2_y}{n}} \right)
$$

A one-sided confidence bound can be obtained by replacing $t_{\alpha/2, \nu}$ with $t_{\alpha, \nu}$. 
The **two-sample t test** for testing $H_0 : \mu_X - \mu_Y = \Delta_0$ is as follows:

Test statistic value: $t = \frac{(\bar{x} - \bar{y}) - \Delta_0}{\sqrt{s^2_x/m + s^2_Y/n}}$

**Alternative Hypothesis**

- $H_a : \mu_X - \mu_Y > \Delta_0$
- $H_a : \mu_X - \mu_Y < \Delta_0$
- $H_a : \mu_X - \mu_Y \neq \Delta_0$

**Rejection Region for Approximate Level $\alpha$ Test**

- $H_a : \mu_X - \mu_Y > \Delta_0$  \( t \geq t_{\alpha,\nu} \) (upper-tailed)
- $H_a : \mu_X - \mu_Y < \Delta_0$  \( t \leq t_{\alpha,\nu} \) (lower-tailed)
- $H_a : \mu_X - \mu_Y \neq \Delta_0$  \( t \geq t_{\alpha/2,\nu} \) or \( t \leq t_{\alpha/2,\nu} \) (two-tailed)
Analysis of Paired Data

Assumptions: The data consists of $n$ independently selected pairs $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$, with $E[X_i] = \mu_1$ and $E[Y_i] = \mu_2$. 

Let $D_1 = X_1 - Y_1$, $D_2 = X_2 - Y_2$, \ldots, $D_n = X_n - Y_n$, so the $D_i$s are the differences within pairs. Then the $D_i$s are assumed to be normally distributed with mean value $\mu_D$ and variance $\sigma^2_D$ (this is usually a consequence of the $X_i$s and $Y_i$s themselves being normally distributed.)
Analysis of Paired Data

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**Paired t Test**

The **paired t test** for testing $H_0 : \mu_D = \Delta_0$ (where $D = X - Y$ is the difference between the first and second observations within a pair, and $\mu_D = \mu_1 - \mu_2$) is as follows:

$$t = \frac{\bar{d} - \Delta_0}{s_D / \sqrt{n}}$$

(where $\bar{d}$ and $s_D$ are the sample mean and standard deviation, respectively, of the $d_i$s)

**Alternative Hypothesis**

<table>
<thead>
<tr>
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</table>
Confidence Interval for $\mu_D$

The **Paired t CI for** $\mu_D$ with confidence level $100(1 - \alpha)\%$ is

$$
\left(\bar{d} - t_{\alpha/2,n-1} \cdot \frac{s_D}{\sqrt{n}}, \quad \bar{d} + t_{\alpha/2,n-1} \cdot \frac{s_D}{\sqrt{n}}\right)
$$

A one-sided confidence bound results from retaining the relevant sign and replacing $t_{\alpha/2,n-1}$ by $t_{\alpha,n-1}$. 
Example: (Problem 40)

Lactation promotes a temporary loss of bone mass to provide adequate amounts of calcium for milk production. The paper “Bone Mass Is Recovered from Lactation to Postweaning in Adolescent Mothers with Low Calcium Intakes” (Amer. J. Clinical Nutr., 2004: 1322-1326) gave the following data on total body bone mineral content (TBBMC) (g) for a sample both during lactation (L) and in the postweaning period (P).

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<th></th>
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<th>3</th>
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Analysis of Paired Data

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Does the data suggest that true average total body bone mineral content during postweaning exceeds that during lactation by more than 25g?
Analysis of Paired Data

The differences for the sample are:

$$-198 \quad -336 \quad -70 \quad -18 \quad -122 \quad -9 \quad -50 \quad -5 \quad -163 \quad -86$$

with mean $\bar{d} = -105.7$ and standard deviation $s_D = 103.8$
Analysis of Paired Data

The differences for the sample are:

\[-198 \quad -336 \quad -70 \quad -18 \quad -122 \quad -9 \quad -50 \quad -5 \quad -163 \quad -86\]

with mean \( \bar{d} = -105.7 \) and standard deviation \( s_D = 103.8 \).

The Quantile-Quantile for this sample differences is

![QQ Plot of Sample Data versus Standard Normal](image)
Proposition

Let $X \sim \text{Bin}(m, p_1)$ and $Y \sim \text{Bin}(n, p_2)$ with $X$ and $Y$ independent. Then

$$E[\hat{p}_1 - \hat{p}_2] = p_1 - p_2$$

so $\hat{p}_1 - \hat{p}_2$ is an unbiased estimator of $p_1 - p_2$. Here $\hat{p}_1 = \frac{X}{m}$ and $\hat{p}_2 = \frac{Y}{n}$.

Furthermore,

$$V[\hat{p}_1 - \hat{p}_2] = \frac{p_1 q_1}{m} + \frac{p_2 q_2}{n}$$

where $q_i = 1 - p_i$. 

Difference Between Population Proportions

Large-Sample Test Procedure

Null hypothesis: $H_0 : p_1 - p_2 = 0$

Test statistic value (large samples):

$$z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p} \hat{q} \left( \frac{1}{m} + \frac{1}{n} \right)}}$$

where $\hat{p} = \frac{X+Y}{m+n} = \frac{m}{m+n} \hat{p}_1 + \frac{n}{m+n} \hat{p}_2$.

Alternative Hypothesis

Rejection Region for Approximate Level $\alpha$ Test

- $H_a : p_1 - p_2 > 0$ \quad $z \geq z_\alpha$ (upper-tailed)
- $H_a : p_1 - p_2 < 0$ \quad $z \leq z_\alpha$ (lower-tailed)
- $H_a : p_1 - p_2 \neq 0$ \quad $z \geq z_{\alpha/2}$ or $z \leq z_{\alpha/2}$ (two-tailed)
Difference Between Population Proportions

Large-Sample Confidence Interval for $p_1 - p_2$

The 100$(1 - \alpha)$% confidence interval for $p_1 - p_2$ is given by

$$
\left( \hat{p}_1 - \hat{p}_2 \right) - z_{\alpha/2} \sqrt{ \frac{\hat{p}_1 \hat{q}_1}{m} + \frac{\hat{p}_2 \hat{q}_2}{n} }, \quad \left( \hat{p}_1 - \hat{p}_2 \right) + z_{\alpha/2} \sqrt{ \frac{\hat{p}_1 \hat{q}_1}{m} + \frac{\hat{p}_2 \hat{q}_2}{n} }
$$
Example: (Problem 50)

Do teachers find their work rewarding and satisfying? The article “Work-Related Attitudes” (Psychological Reports, 1991: 443-450) reports the results of a survey of 395 elementary school teachers and 266 high school teachers. Of the elementary school teachers, 224 said they were very satisfied with their jobs, whereas, 126 of the high school teachers were very satisfied with their work. Is there any difference between the proportion of all elementary school teachers who are satisfied and all high school teachers who are satisfied?