RESEARCH STATEMENT

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1. Overview

My research is focused on nonlinear analysis and nonlinear partial differential equations. In particular, under the supervision of Professor Klaus Schmitt, I am concerned with studying boundary value problems involving the $p$–Laplace operator and some of its generalizations, where the $p$–Laplace operator is given by

$$\Delta_p = \text{div}(\nabla |\nabla|^{p-2})$$

$p > 1$.

Specifically, I study problems of the form

$$(1.1) \quad L(u) = F(u),$$

where $L$ is such a differential operator acting on a subspace of the Sobolev space $W^{1,p}(\Omega)$ ($\Omega$ is a bounded domain in $\mathbb{R}^N$) and $F$ is a functional satisfying various conditions. Depending on the nature of the functional $F$, my research may be described as follows:

1. **On positive solutions of quasilinear elliptic equations.** My first research studied (1.1) when $F$ is given by

$$F(u) = f(u(\cdot)), \quad u \in W^{1,p}_0(\Omega),$$

where $f$ is a continuous function defined on $\mathbb{R}$ and $L = -\Delta_p$. The results obtained (see [33]) extend the papers [10, 22] in the sense that the Laplace operator in these two papers can be replaced with its generalizations, the $p$–Laplace operator. Also, our results can be applied to singular elliptic equations.

2. **Boundary value problems for singular elliptic equations.** My second project studies (1.1) when $L = -\Delta_p$ and $F$ is given by

$$F(u)(x) = a(x)g(u(x)) + \lambda h(x, u(x))$$

where $a \in L^\infty(\Omega)$ and $\lambda \geq 0$ is a parameter. The function $g$ is called the singular term because $g(x, \cdot)$ is allowed to blow up as $u \to 0^+$, for a.e. $x \in \Omega$ and $\lambda h$ is called the parameter dependent term. In this project, we establish a sub-supersolution theorem and use an eigenfunction of the $p$–Laplacian to construct sub- and super-solutions to these problems. Our assumptions on the singular term $g$ are more relaxed than in some previous papers, even for the case $p = 2$, as $g$ might not be necessarily monotone. We also study how the growth of $h$ affects our existence result.
3. **Elliptic singular problems with convection terms.** In this project, we consider the existence results for a class of both nonsingular and singular problems when \( L \) is an elliptic (\( S_+ \)) operator and \( F(u) = f(\cdot, u, \nabla u) \). The results obtained were motivated by several results obtained for the case \( p = 2 \) and allow for more general nonlinearities, even for the case when \(-L\) is the Laplacian.

4. **Nonlocal problems modelling shear banding.** In this project, we employed a singularity result of Giacomoni, Schindler and Takáč [19] and Leray-Schauder continuation arguments (see [12, 37]) to find an unbounded continuum for the parameter dependent boundary value problem of the form (1.1) when \( L = -\Delta_p \) and

\[
F(u) = \frac{\lambda f(\cdot, u)}{(\int_{\Omega} g(y,u)dy)^r},
\]

where \( f \) and \( g \) are two continuous functions. The result obtained can be applied to the Liouville-Bratu-Gelfald problem

\[
\begin{aligned}
-\Delta_p u &= \lambda e^u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

and the nonlocal problem modelling shear banding

\[
\begin{aligned}
-\Delta_p u &= \frac{\lambda e^u}{(\int_{\Omega} e^u dy)^r} \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega, \quad r > 0.
\end{aligned}
\]

2. **Description of results**

2.1. **On positive solutions of quasilinear elliptic equations.** In 1981, Peter Hess [22] established a multiple existence result for

\[
\begin{aligned}
-\Delta u &= \lambda f(u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \lambda \) is a nonnegative parameter and \( f \) is a continuous function on \( \mathbb{R} \) and satisfies:

1. \( f(0) \geq 0 \),
2. there exist \( a_1 < b_1 < a_2 < b_2 < \cdots < b_{m-1} < a_m, m \geq 2 \), such that for all \( k = 1, \cdots, m-1 \)
   \[
   \begin{aligned}
   f(\cdot) &\leq 0 \quad \text{on } (a_k, b_k) \\
   f(\cdot) &\geq 0 \quad \text{on } (b_k, a_{k+1})
   \end{aligned}
   \]

(See e.g. figure 1). Hess proved that if the function \( f \) satisfies

\[
\int_{a_k}^{a_{k+1}} f(s)ds > 0
\]

for all \( k \in \{1, \cdots, m-1\} \), then for all \( \lambda \), sufficiently large, (2.1) has at least \( m-1 \) non-negative solutions

\[
\{u_1, \cdots, u_{m-1}\} \subset W^{1,p}_0(\Omega) \cap L^\infty(\Omega)
\]
such that
\[ a_k < \| u_k \|_{\infty} \leq a_{k+1} \]
for each \( k = 1, \cdots, m - 1 \). Later, Dancer and Schmitt [10] proved the converse of this result; namely, if (2.1) has a solution \( u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) such that
\[ a_k < \| u \|_{\infty} \leq a_{k+1} \]
for some \( k \in \{1, \cdots, m - 1\} \) then \( f \) must satisfy
\begin{equation}
\int_{a_k}^{a_{k+1}} f(s)ds > 0.
\end{equation}

Motivated by these two results, we published [33] and proved that when \( \Delta \) is replaced by \( \Delta_p \), the conclusions of Hess [22] and Dancer and Schmitt [10] are still guaranteed. More precisely, the problem
\[
\begin{cases}
-\Delta_p u = \lambda f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
has a solution \( u \) in \( W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) such that
\[ a_k < \| u \|_{\infty} \leq a_{k+1} \]
for some \( k \in \{1, 2, \cdots, m - 1\} \) if, and only if,
\[ \int_{a_k}^{a_{k+1}} f(s)ds > 0. \]

In order to obtain the results above, we establish a strong maximum principle which is similar to but independent of that of Vazquéz [41]. Indeed, we remove the requirement on the solution \( u \) that \( \Delta u \in L^2_{\text{loc}}(\Omega) \) in Vazquéz’ paper. Then, we use this maximum principle,
variational methods and sub-supersolution theorems in [27, 28] to prove our main theorems. Note that the condition \( f(0) \geq 0 \) may be removed by using sub-supersolution theorems of Le and Schmitt in [27, 28] again. Moreover, the obtained results in can be used to study infinite semipositone elliptic problems; i.e, requiring that \( f(0) = -\infty \).

This work has been published in [33].

2.2. **Boundary value problems for singular elliptic equations.** We study singular elliptic equations in this project. Such problems arise in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogenous catalysts, in the theory of heat conduction in electrically conducting materials and therefore have been intensively studied in the last decades. The following is the problem we are interested in:

\[
\begin{cases}
-\Delta_p u = \alpha g(u) + \lambda h(u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \lambda \) is a nonnegative parameter;

\[ a : \Omega \to [1, \infty) \]

is in \( L^\infty(\Omega) \);

\[ g : (0, \infty) \to [0, \infty) \]

is continuous and satisfies

\[ \lim_{s \to 0^+} g(s) = \infty; \]

and,

\[ h : [0, \infty) \to \mathbb{R} \]

is continuous.

Two pioneering papers regarding such singular problems are those of Lazer and McKenna [26] and Crandall, Rabinowitz and Tartar [9]. These papers motivated a flurry of work in subsequent years (see, e.g., [6, 8, 19, 25, 38, 39, 44, 45]). All of these papers studied (2.3) in the case \( p = 2 \) and under the assumption that the singular term \( g \) either takes a particular form or satisfies a monotonicity condition. Thus the question arises whether or not the existence of solutions for (2.3) is still guaranteed when \( p \in (1, \infty) \) and the monotonicity property is removed. Hai [20, 21], has given affirmative answers to this question in the case that \( \Omega \) is an annulus, by establishing existence results for radial solutions which are solutions of associated ordinary differential equations.

We approach problem (2.3) by proving a version of a sub-supersolution theorem for singular elliptic problems and then finding such a well-ordered pair of sub-supersolutions for the specific singular problem under consideration. With this method, we can allow \( p \in (1, \infty) \) and remove not only the monotonicity condition but also some technical conditions on the singular terms in the papers above.

Our main result can be summarized as follows.
Assume $g$ satisfies:

$$\exists \gamma > 0, \ C > 0 \text{ such that } g(s) \leq Cs^{-\gamma}, \ \forall s \in (0, \infty).$$

Then:

(i) if $\limsup_{s \to 0^+} \frac{h(s)}{s^{p-1}} < \infty$, there exists $\tilde{\lambda} > 0$ such that for all $\lambda \in [0, \tilde{\lambda}]$, problem (2.3) has a solution,

(ii) if there exists $\alpha < p - 1$ such that

$$0 \leq h(s) \leq s^\alpha, \ \forall s \in [1, \infty),$$

then for all $\lambda \geq 0$, problem (2.3) has a solution.

An interesting fact about the singular problem (2.3) is that when $\gamma \geq \frac{2p-1}{p-1}$, the solution obtained by the result above is not in $W_0^{1,p}(\Omega)$, which is usually true for $p$–Laplace equations. For example, in the case $N = 1$ and $\Omega = (0, 1)$, the function $u$ defined by

$$x \mapsto \sqrt{2x(1-x)}$$

does not belong to $W_0^{1,2}(0, 1)$ and is the unique solution of the boundary value problem

$$-u'' = u^{-3} \text{ in } (0, 1), \ u(0) = u(1) = 0.$$

The details of this project are provided in [30].

2.3. Elliptic singular problems with convection terms. In [31], we consider the following problem

$$\begin{cases} -\Delta_p u = f(x, u) + g(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases}$$

where $f$ and $g$ are Carathéodory functions defined on $\Omega \times \mathbb{R}$ and $\Omega \times \mathbb{R} \times \mathbb{R}^N$, respectively. Moreover, $f$ is allowed to be singular in the following sense:

$$\lim_{s \to 0} f(x, s) = \infty \text{ uniformly in } x \in \Omega.$$

This problem is interesting because it arises from some physical phenomena (see [17]) and was studied in several seminal papers, e.g., [1, 17, 18].

When seeking a weak solution of (2.5), i.e., looking for a function $u \in W_0^{1,p}(\Omega)$ such that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx = \int_{\Omega} [f(x, u) + g(x, u, \nabla u)] v dx$$

for all $v \in W_0^{1,p}(\Omega)$, one considers the operator $\Delta_p$ as the mapping

$$\Delta_p : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*,$$

defined by

$$\langle \Delta_p u, v \rangle = -\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx \quad \text{for all } u, v \in W_0^{1,p}(\Omega).$$
With this definition, \(-\Delta_p\) is of class \((S_+)\) (see [13]).

The difficulties in the study of problems of the form (2.5) are the singular behavior of \(f\) and the gradient dependence of term \(g\), which we shall call the convection term. In the presence of a convection term, variational methods often fail, because of the lack of a corresponding energy functional.

If we consider the mapping
\[
T : W_0^{1,p}(\Omega) \to \left( W_0^{1,p}(\Omega) \right)^*,
\]
defined by
\[
\langle Tu, v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx - \int_\Omega [f(x,u) - g(x,u,\nabla u)v] dx,
\]
for all \(u, v \in W_0^{1,p}(\Omega)\), then, by imposing suitable growth conditions on \(f\) and \(g\), the mapping \(T\) can be shown to belong to class \((S_+)\) and, therefore, its topological degree can be defined (see Browder [5]). Hence topological degree arguments may be employed. The difficulties caused by the singular behavior of \(f\) will be overcome by approximating the desired solution by solutions \(u_n\) of
\[
\begin{aligned}
-\Delta_p u_n &= f(x,u_n + n^{-1}) + g(x,u_n,\nabla u_n) \quad \text{in } \Omega, \\
  u_n &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
n \(\in\mathbb{N} := \{1, 2, 3, \ldots\}\).

The following is an outline of the project. We prove first that problems of the form
\[
\begin{aligned}
-\Delta_p u &= G(x,u,\nabla u) \quad \text{in } \Omega, \\
  u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where \(G\) is a Carathéodory function defined on \(\Omega \times \mathbb{R} \times \mathbb{R}^N\), have at least a weak solution provide that the function \(G\) satisfies certain growth conditions. The result obtained is more general than that in [11], even in the case \(p = 2\), in the sense that we no longer require a certain Lipschitz continuity of \(G\), which was assumed in [11] to be able to employ the iterative technique
\[
u_n = (-\Delta_p)^{-1}(G(x,u_n,\nabla u_{n-1})), \quad n \geq 2,
\]
similar to the proof of the contraction principle. Our result also extends those of Xavier [42] and Yan [43], since we no longer require a dependence of \(p\) on the dimension \(N\). Secondly, we consider singular elliptic problems of the form (2.5). Our existence result there is motivated by the work of Alves, Carrião and Faria [1], where, for the case \(p = 2\), a Galerkin scheme was employed. This scheme required local Hölder continuity requirements on \(f\) and \(g\), which can be dispensed with in our work.

2.4. Nonlocal problems modelling shear banding. One of the difficulties in studying problems (2.3) and (2.5) is the fact that if \(u \in W_0^{1,p}(\Omega)\) then, when \(x\) is close to \(\partial \Omega\), \(g(u(x))\)
and $f(x, u(x))$ blow up and, therefore, these two functions might not be in $(W_0^{1,p}(\Omega))^*$. More generally, we might understand that the problem

\[
\begin{aligned}
-\Delta_p u &= f(x, u) \quad \text{in } \Omega, \\
    u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]

is singular if $f(\cdot, v(\cdot))$ might not be in $(W_0^{1,p}(\Omega))^*$ for some function $v \in W_0^{1,p}(\Omega)$. In particular, the Liouville-Gelfand-Bratu problem

\[
(2.6)
\begin{aligned}
-\Delta_p u &= \lambda e^u \quad \text{in } \Omega, \\
    u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]

where $\lambda$ is a nonnegative parameter, is singular (in dimension greater than 2). When $\Omega$ is a ball, this problem was considered in many papers (see e.g. [4, 7, 23, 24, 29]) and the structure of radially symmetric solutions in the $\lambda - u$ plane was completely described in [24] when $p = 2$ and in [23] for $p \in (1, \infty)$. Another singular crucial model for shear banding, looking similar to but more general than (2.6), is the following nonlocal problem

\[
(2.7)
\begin{aligned}
-\Delta_p u &= \frac{\lambda e^u}{(\int_{\Omega} e^{h(y, u)} dy)} \quad \text{in } \Omega, \\
    u &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
\]

It was first introduced and studied by Berbernes and Talaga [3] and then again by Miyasita [36] in 2007. They also described the structure of solutions on $\lambda - u$ plane in the case $\Omega$ is a ball. A question arises whether or not problems (2.6) and (2.7) are solvable when $\Omega$ might not necessarily be a ball. We obtained results giving an affirmative answer. In particular, we are interested in solving the nonlocal problem

\[
(2.8)
\begin{aligned}
-\Delta_p u &= \frac{\lambda f(x, u)}{(\int_{\Omega} h(y, u) dy)} \quad \text{in } \Omega, \\
    u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]

of which (2.6) and (2.7) are two particular cases, where $r \geq 0$ and $\Omega$ is smooth bounded domain of $\mathbb{R}^N$ and might not necessarily be a ball. In details, we use Theorem B1 in [19] to show that the map

\[
L_\lambda : C^1(\overline{\Omega}) \cap W_0^{1,p}(\Omega) \to C^1(\overline{\Omega}) \cap W_0^{1,p}(\Omega)
\]

\[
u \mapsto (-\Delta_p)^{-1} \left( \frac{\lambda f(x, u)}{(\int_{\Omega} h(y, u) dy)^r} \right)
\]

is completely continuous. Then employing Leray-Schauder continuation arguments (see [12, 37]), we conclude that problem

\[
u - L_\lambda \nu = 0,
\]

has an unbounded continuum of solutions in $C \subset [0, \infty) \times C^1(\overline{\Omega})$.

The details about this result are given in [32].

Besides the work, described above, I published, together with colleagues, the following papers during the time I worked at Vietnam National University at Hochiminh City. They are [14, 15, 16].
3. Future plans

One of my future plans is to extend our result in [31]. We believe that the result in [31] can be guaranteed if the growth condition on the convection term is controlled by the power $p$ as in the paper of Mawhin and Schmitt [34] when $p = 2$ and the book by Troianiello [40]. My method of attack will be to establish a suitable sub-supersolution theorem and then use it to solve the considered problem.

Another interesting mathematical problem is how to extend a function $f$, defined and Lipschitz continuous on $\partial\Omega$, so that $f$ is still Lipschitz continuous, with the same Lipschitz constant, and defined on the whole region $\Omega$. This problem plays a crucial role in image processing; especially, in recovering a damaged image. Aronsson [2] provided an answer to this problem by solving

$$
\begin{align*}
-\Delta_\infty u &= 0 \quad \text{in } \Omega, \\
u &= f \quad \text{on } \partial\Omega,
\end{align*}
$$

where

$$
\Delta_\infty = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}
$$

is the $\infty-$Laplacian. The $\infty-$Laplacian can be considered as the limit of $p-$Laplacian when $p$ approaches $\infty$. I am therefore motivated to see whether my results described above may be used to obtain existence results about boundary value problems involving the infinity Laplacian.

Besides working with my advisor, Professor Schmitt, I studied applied mathematics with Professor Milton in my last year as a student at the University of Utah because I really want to learn mathematics from the viewpoint of physics. He asked me to read his book [35] from which I have learned how variational principles may be used to study composite materials and related subjects. During the coming semester I shall be supported by a research fellowship working with Professor Milton and thus hope to deepen my understanding of variational methods as used in the study of composite materials.

References


[29] L. Liouville, *Sur l’équation aux différences partielles $d^2 \log \lambda \over \partial u^2 \pm \lambda a^2 = 0$*, J. Math. Pures Appl, 18 (1853), pp. 71–72.


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