Exercise 2.28 There are several ways to do this. First notice that what we want is the conditional probability of $O_1 = 0, O_2 = 1$ given that $O_1 \neq O_2$, since we only consider the outcomes of the two tosses when $O_1 \neq O_2$. So we want compute:

\[
\mathbb{P}\{O_1 = 0, O_1 = 1 \mid O_1 \neq O_2\} = \frac{\mathbb{P}\{O_1 = 0, O_1 = 1 \text{ and } O_1 \neq O_2\}}{\mathbb{P}\{O_1 \neq O_2\}} = \frac{p(1-p)}{2p(1-p)} = \frac{1}{2}.
\]

Alternatively, we can reason as follows: Let $T$ be the number of trials required before the two coins come up differently.

\[
\mathbb{P}\{X = 1\} = \sum_{n=1}^{\infty} \mathbb{P}\{X = 1 \text{ and } T = n\} = \sum_{m=0}^{\infty} (p^2 + (1-p)^2)^m p(1-p) = \frac{p(1-p)}{1-(p^2 + (1-p)^2)} = \frac{1}{2}.
\]

The procedure in (b) is to flip a coin until the last flip is different from the next to last flip. Equivalently, flip until the coin is different from the first flip. With probability $p$ this will be tails, and with probability $1 - p$ this will be heads.

Exercise 2.34 $f(x)$ must integrate to 1, so

\[
\int_{0}^{2} c(4x - 2x^2)dx = c(\frac{2x^2}{3} - \frac{2}{3}x^3)|_0^2 = c(2 - \frac{8}{3}) = \frac{8}{3}.
\]

Thus $c = \frac{3}{8}$, and

\[
f(x) = \frac{3}{2}x - \frac{3}{4}x^2.
\]
\[ P \left\{ \frac{1}{2} < X < \frac{3}{2} \right\} = \int_{1/2}^{3/2} \left( \frac{3}{2}x - \frac{3}{4}x^2 \right) \, dx \]
\[ = \frac{3}{4}x^2 - \frac{1}{4}x^3 \bigg|_{1/2}^{3/2} \]
\[ = \frac{5}{4} \]

Exercise 2.37

\[ P \left\{ \max_{1 \leq k \leq n} X_k \leq x \right\} = P \{ X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x \} \]
\[ = P \{ X_1 \leq x \} \cdots P \{ X_n \leq x \} \]
\[ = x^n. \]

Since the distribution function is \( F_M(x) \), the density is given by
\[ f_M(x) = \frac{d}{dx} F_M(x) = nx^{n-1}. \]

Exercise 2.43

\[ X = \sum_{i=1}^{n} X_i. \]

Thus

\[ E \{ X \} = \sum_{i=1}^{n} E \{ X_i \} \]
\[ = nP \{ \text{red ball 1 is take before a black ball} \} \]

The full solution will be provided later. See the current computer homework assignment.

Example 2.44

Let
\[ Y_i = \begin{cases} 
1 & \text{if red ball } i \text{ is chosen between first and second black ball drawn} \\
0 & \text{otherwise} 
\end{cases} \]
As before, then

\[ \mathbb{E}\{Y\} = n \mathbb{P}\{\text{Red ball 1 is between first and second black balls}\} . \]

The full solution is part of the next computer HW.

Exercise 2.47

\[ \mathbb{E}\{X^2\} = \int_{0}^{1} x^2 \, dx = \left. \frac{1}{3} x^3 \right|_{0}^{1} = \frac{1}{3} . \]

Exercise 2.53

\[ \mathbb{E}\{X^n\} = \int_{0}^{1} x^n \, dx = \left. \frac{1}{n+1} x^{n+1} \right|_{0}^{1} = \frac{1}{n+1} . \]

Also,

\[ \mathbb{E}\{(X^n)^2\} = \int_{0}^{1} x^{2n} \, dx = \frac{1}{2n+1} . \]

Thus,

\[ \text{Var}(X^n) = \mathbb{E}\{(X^n)^2\} - (\mathbb{E}\{X^n\})^2 \]

\[ = \frac{1}{2n+1} - \left( \frac{1}{n+1} \right)^2 . \]

Exercise 2.57 \(X\) is the sum of \(n\) Bernoulli\((p)\) r.v.s, and \(Y\) is the sum of \(m\) Bernoulli\((p)\) r.v.s, and so \(X + Y\) is the sum of \(n + m\) Bernoulli\((p)\) r.v.s.

Exercise 2.67

\[ \mathbb{P}\{5 < X < 15\} = \mathbb{P}\{-5 < X - 10 < 5\} \]

\[ = \mathbb{P}\{(X - 10)^2 < 25\} \]

\[ \leq \frac{\text{Var}(X)}{25} \]

\[ = \frac{3}{5} . \]
Exercise 3.12 The marginal of $Y$ is given by

$$f_Y(y) = \int_0^\infty \frac{e^{-x/y}e^{-y}}{y} \, dx$$

$$= \frac{e^{-y}}{y} \int_0^\infty e^{-x/y} \, dx$$

$$= \frac{e^{-y}}{y} ye^{-x/y} \bigg|_0^\infty$$

$$= \frac{e^{-y}}{y}.$$ 

Thus the conditional density of $X \mid Y = y$ is given by

$$\frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1}{y} e^{-x/y}.$$

That is to say, given $Y = y$, the distribution of $X$ is exponential($1/y$). Thus it has conditional expectation $y$.

Exercise 3.14 Notice that for $u \leq 1/2$,

$$\mathbb{P}\left\{ U \leq u \mid U \leq \frac{1}{2} \right\} = \frac{\mathbb{P}\{ U \leq u \text{ and } u \leq \frac{1}{2} \}}{\mathbb{P}\{ U \leq \frac{1}{2} \}}$$

$$= 2 \mathbb{P}\{ U \leq u \}$$

$$= 2u.$$

Thus the density of $U$ conditioned on $U < 1/2$ is

$$f(u) = \begin{cases} 
2 & \text{if } 0 < u < 1/2 \\
0 & \text{otherwise.}
\end{cases}$$

And so

$$\mathbb{E}\left\{ U \mid U < \frac{1}{2} \right\} = \int_0^{1/2} 2udu$$

$$= u^2 \bigg|_0^{1/2}$$

$$= \frac{1}{4}.$$
Exercise 3.30

\[ \mathbb{E}\{N\} = \sum_{j=1}^{m} \mathbb{E}\{N \mid X_0 = j\} p(j) \]
\[ = \sum_{j=1}^{m} \frac{1}{p(j)} p(j) \]
\[ = m. \]

Thus follows because conditional on \(X_0 = j\), the distribution of \(N\) is geometric(\(p(j)\)).

Exercise 3.36

\[ \mathbb{E}\{N\} = \mathbb{E}\{\mathbb{E}\{N \mid U\}\} \]
\[ = \mathbb{E}\{nU\} \]
\[ = \frac{n}{2}, \]
\[ \mathbb{E}\{N^2\} = \mathbb{E}\{\mathbb{E}\{N^2 \mid U\}\} \]
\[ = \mathbb{E}\{nU(1 - U) + (nU)^2\} \]
\[ = \mathbb{E}\{nU + n(n - 1)U^2\} \]
\[ = \frac{n}{2} + n(n - 1)\frac{1}{3} \]
\[ \text{Var}(N) = \frac{n}{2} + n(n - 1)\frac{1}{3} - \frac{n^2}{4} \]
\[ = \frac{1}{6}n + \frac{1}{12}n^2. \]

Exercise 3.42 If \(A\) is the total amount spent, then \(N = \sum_{i=1}^{N} U_i\) where \(\{U_i\}\) is an i.i.d. sequence of Uniform[0, 100] r.v.s., and \(N\) is Poisson(10).

Thus

\[ \mathbb{E}\{A\} = \mathbb{E}\left\{\mathbb{E}\left\{\sum_{i=1}^{N} U_i \mid N\right\}\right\} \]
\[ = \mathbb{E}\{50N\} \]
\[ = 500. \]

For the variance, see page 112 of the text.