3 More on fair games

**Definition 3.1.** Let \( \{X_n\}_{n=0}^\infty \) be a stochastic process. A *fair game* (called a *martingale* in the probability literature) with respect to \( \{X_n\}_{n=0}^\infty \) is a stochastic process \( \{M_n\}_{n=0}^\infty \) so that

(i) there exist a sequence of functions \( \{g_n\}_{n=0}^\infty \) so that \( M_n = g_n(X_0, \ldots, X_n) \) for all \( n \),

(ii) \( \mathbb{E}\{M_{n+1} | X_0, \ldots, X_n\} = M_n \).

We can rewrite condition (ii) as

\[
\mathbb{E}\{M_{n+1} - M_n | X_0, \ldots, X_n\} = 0, \tag{1}
\]

which is sometimes a more convenient form. Indeed, we can write

\[
M_n = M_0 + \sum_{k=1}^{n} \Delta M_k \tag{2}
\]

where \( \Delta M_k \overset{\text{def}}{=} M_{k+1} - M_k \). The property (1) says that \( \mathbb{E}\{\Delta M_k | X_0, X_1, \ldots, X_k\} = 0 \) for all \( k \). Thus (2) and (1) say that a fair game is the sum of its initial value plus increments which have expectation 0 *conditional on the past.*

We may suppose that \( \Delta M_k \) is the amount we either win or lose at time \( k \), the random variable \( M_0 \) is our initial fortune, and \( M_n \) is our fortune at time \( n \). Equation (2) represents our current fortune as our initial fortune plus the sum of our winnings in a series of successive games. (A negative win is a loss.) The key property is that conditional on what has transpired up to time \( k \), i.e. on the random variables \( X_0, \ldots, X_k \), our expected gain on the \( (k+1) \)st game (that is, \( \Delta M_k \)) is zero.

**Example 3.2.** Let \( D_1, D_2, \ldots \) be a sequence of independent random variables so that \( \mathbb{E}\{D_k\} = 0 \) for all \( k \). If \( M_n \overset{\text{def}}{=} \sum_{k=1}^{n} D_k \), then

\[
\mathbb{E}\{M_{n+1} - M_n | D_1, \ldots, D_n\} = \mathbb{E}\{D_{n+1} | D_1, \ldots, D_n\} = 0,
\]

so \( \{M_n\} \) is a fair game. (It is clear that \( M_n \) is a function of \( D_1, \ldots, D_n \), so condition (i) of Definition 3.1 is satisfied.)
Example 3.3. Let \( \{X_n\}_{n=0}^{\infty} \) be a random walk which moves up one unit with probability \( p \), and down one unit with probability \( 1 = 1 - p \), where \( p \neq 1/2 \). In other words, given \( X_0, \ldots, X_n \),
\[
\Delta X_n \overset{\text{def}}{=} X_{n+1} - X_n = \begin{cases} 
1 & \text{with probability } p \\
-1 & \text{with probability } q .
\end{cases}
\]
If \( M_n \overset{\text{def}}{=} (q/p)^X_n \), then \( \{M_n\}_{n=0}^{\infty} \) is a martingale with respect to \( \{X_n\}_{n=0}^{\infty} \). Condition (i) is clear,
and
\[
\mathbb{E}\{(q/p)^{X_{n+1}} \mid X_0 = x_0, \ldots, X_n = x_n\} = \mathbb{E}\{(q/p)^{x_n}(q/p)^{X_{n+1}-x_n} \mid X_0 = x_0, \ldots, X_n = x_n\}
= (q/p)^{x_n}\left(p(q/p) + q(q/p)^{-1}\right)
= (q/p)^{x_n}.
\]

Example 3.4. Let \( \{X_n\}_{n=0}^{\infty} \) be as in the previous example. Let \( \mu \overset{\text{def}}{=} p - q \), and \( M_n \overset{\text{def}}{=} X_n - \mu n \). Then
\[
\mathbb{E}\{M_{n+1} - M_n \mid X_0, \ldots, X_n\} = p - q - \mu
= 0,
\]
so \( \{M_n\} \) is a fair game.

Definition 3.5 (Strategy). A \textit{strategy} is a sequence of random variables \( \{B_n\}_{n=0}^{\infty} \) so that \( B_n = f_n(X_0, \ldots, X_{n-1}) \) for some function \( f_n \) of \( n \) variables.

The point is that \( B_n \) is completely determined by what has happened strictly before time \( n \), that is, by \( X_0, \ldots, X_{n-1} \).

Suppose that \( \{M_n\}_{n=0}^{\infty} \) is a martingale with respect to some stochastic process \( \{X_n\} \), and \( \{B_n\} \) is a strategy. We can think of \( B_n \) as the amount we are willing to stake on a game which pays out \( M_{n+1} - M_n \), after observing \( X_0, X_1, \ldots, X_n \). In this case, the amount that we win on this game will be
\[
B_n \Delta M_n = B_n(M_{n+1} - M_n).
\]
The case $B_n = 0$ corresponds to a decision not to play the game, or in other words, to wager nothing. If we play $n$ times, using the strategy $\{B_n\}$, then our total fortune will be

$$M_0 + \sum_{k=1}^{n} B_k \Delta M_{k-1} = M_0 + \sum_{k=1}^{n} B_k(M_k - M_{k-1}). \quad (3)$$

The point in the definition of a strategy is that we are not permitting a scheme in which we are clairvoyant (able to see into the future). For example, we would like to use the scheme that $B_n = 1$ if $M_n - M_{n-1} > 0$ and $B_n = 0$ otherwise. Then we would only bet on the games which we win, and we would only increase our fortune. We don’t allow such a strategy as it presumes that we know the outcome of a game before we decide whether or not to bet.

Equation (3) motivates us to make the definition

$$(B \circ M)_n \overset{\text{def}}{=} M_0 + \sum_{k=0}^{n-1} B_k(M_k - M_{k-1}),$$

where $\{M_n\}$ is a fair game, and $\{B_n\}$ is a strategy.

The next result shows that if we are not allowed to look into the future, any strategy will still produce a fair game.

**Theorem 3.6.** For any strategy $\{B_n\}_{n=0}^\infty$, the sequence of random variables $\{(B \circ M)_n\}_{n=0}^\infty$ is a fair game.

**Proof.**

$$\mathbb{E} \{(B \circ M)_{n+1} - (B \circ M)_n \mid X_0, \ldots, X_n\} = \mathbb{E} \{B_{n+1}(M_{n+1} - M_n) \mid X_0, \ldots, X_n\} = B_{n+1} \mathbb{E} \{M_{n+1} - M_n \mid X_0, \ldots, X_n\} = 0.$$

We are allowed to move $B_{n+1}$ outside of the conditional expectation because $X_0, \ldots, X_n$ determine the value of $B_{n+1}$, and so conditional on $X_0, \ldots, X_n$, the random variable $B_{n+1}$ acts like a constant and can be moved outside the expectation. (Recall that in general if $U, V$ are random vectors, and $g$ is a function, $\mathbb{E} \{g(U)V \mid U\} = g(U)\mathbb{E} \{V \mid U\}$. )
**Definition 3.7.** A stopping time is a random variable $T$ with values in $\{0, 1, \ldots\}$ so that the event $\{T = n\}$ is determined by the random variables $X_0, \ldots, X_n$. That is, there is some function $f_n$ of $n + 1$ variables so that

$$1\{T = n\} = f_n(X_0, \ldots, X_n).$$

**Example 3.8.** Let $T_z = \min\{n \geq 0 : X_n = z\}$ be the first time that $\{X_n\}$ visits state $z$. Then

$$1\{T_z = n\} = \{X_0 \neq z, X_1 \neq z, \ldots, X_{n-1} \neq z, X_n = z\},$$

so $T_z$ is a stopping time.

**Example 3.9.** Let $T = \min\{n \geq m : X_n = \max_{j \leq m} X_j\}$. Thus $T$ is the first time after $m$ that $X_n$ is equal to the maximum of $X_j$ over $j = 0, \ldots, m$. Then for $n < m$ the event $\{T = n\}$ never occurs, and so $1\{T = n\} \equiv 0$, which is a trivial function of $X_0, \ldots, X_n$. For $n \geq m$,

$$\{T = n\} = \{X_m \neq \max_{j \leq m} X_j, \ldots, X_n = \max_{j \leq m} X_j\},$$

and so $1\{T = n\}$ is a function of $(X_0, \ldots, X_n)$. Thus $T$ is a stopping time.

**Example 3.10.** Let $T = \min\{n \geq 0 : X_n = \max_j X_j\}$. Then then event $\{T = n\}$ can be written as $\{X_n = \max_j X_j\}$, and hence depends on all the random variables $X_0, X_1, \ldots$ not just $(X_0, \ldots, X_n)$. Hence it is not a stopping time.

The next results show why stopping times are important:

**Theorem 3.11.** Let $T$ be a stopping time, and $\{M_n\}_{n=0}^\infty$ be a fair game. Then $\{M_{n\wedge T}\}_{n=0}^\infty$ is a fair game.

**Corollary 3.12.** Let $\{M_n\}_{n=0}^\infty$ be a fair game and $T$ a stopping time so that $|M_{n\wedge T}| \leq K$ for all $n$, where $K$ is a fixed number. Then

$$E\{M_T\} = E\{M_0\}.$$
Proof of Theorem 3.11. Let \( B_n = \mathbf{1}\{T > n\} \). Then

\[
B_n = 1 - \mathbf{1}\{T \leq n - 1\} = 1 - \sum_{k=1}^{n-1} \mathbf{1}\{T = k\},
\]

and since \( T \) is a stopping time, \( B_n \) can be written as a function of \( X_0, \ldots, X_{n-1} \). Thus \( \{B_n\}_{n=0}^\infty \) is a strategy. Check that

\[
(B \circ M)_n = M_{T \wedge n} - M_0.
\]

Thus \( M_{T \wedge n} - M_0 \) is a fair game. The reader should check then that \( M_{T \wedge n} - M_0 + M_0 = M_{T \wedge n} \) is still a fair game. \( \blacksquare \)

Proof of Corollary 3.12. Since \( \{M_{T \wedge n}\} \) is a fair game, we have

\[
\mathbb{E}\{M_{T \wedge n}\} = \mathbb{E}\{M_0\}.
\]

Thus

\[
\lim_{n \to \infty} \mathbb{E}\{M_{T \wedge n}\} = \mathbb{E}\{M_0\}.
\]

Since \( M_{T \wedge n} \) is bounded, we are allowed to take a limit inside the expectation. Note that we cannot in general move a limit inside an expectation. In this case, because the random variables inside the expectation are bounded, we are permitted to do so. Thus, \( \mathbb{E}\{M_T\} = \mathbb{E}\{M_0\} \).

\( \blacksquare \)

Example 3.13. Let \( X_0 \equiv 0 \), and let \( X_1, X_2, \ldots \) be a sequence of independent and identically distributed random variables with

\[
\mathbb{P}\{X_k = 1\} = \mathbb{P}\{X_k = -1\} = \frac{1}{2}.
\]

Then \( S_n = \sum_{k=0}^n X_k \) is a fair game. Let \( B_1 \equiv 1 \), and for \( n > 1 \), let

\[
B_n = \begin{cases} 
2^n & \text{if } X_1 = X_2 = \cdots = X_{n-1} = -1 \\
0 & \text{if } X_j = 1 \text{ for some } j < n.
\end{cases}
\]
Thus, provided we have not won a single previous game, we bet $2^n$, and as soon as we win, we stop playing. Then if $T$ is the first time that we win, $T$ is a stopping time.

$$M_n \overset{\text{def}}{=} (B \circ S)_n = \begin{cases} 
0 & \text{if } n = 0, \\
-2^{(n-1)} & \text{if } 1 \leq n < T, \\
1 & \text{if } n \geq T.
\end{cases}$$

We have that $M_T = 1$, since we are assured that eventually $X_n = 1$ and so $T$ is always a finite number. Thus $\mathbb{E} \{M_T\} = 1$. But $\mathbb{E} \{M_0\} = 0$, and $\{M_n\}$ is a fair game! We are playing a fair game, but have allowed that if we stop at a stopping time, we can assure ourselves a profit. This at first glance seems to contradict Corollary 3.12. But notice that the condition $|M_{T \wedge n}| < K$ is not satisfied, so we cannot apply the Corollary.

In “real life”, a gambler has only finite capital, and casinos don’t permit one to gamble on credit. Thus a gambler is only allowed a finite number of losses before he must quit playing (having lost all his capital.) Thus a gambler can only guarantee himself to make the random amount $M_{N \wedge T}$, where $N = \log_2 C + 1$, where $C$ is his initial capital. (This is because after playing $N$ games and not winning, his loss is $-2^{\log_2 C} = -C$, and he must quit.) Since $M_{n \wedge T}$ is a fair game for any fixed $n$, and $N$ is a fixed number, his expected return is $\mathbb{E} \{M_{N \wedge T}\} = 0$.

### 4 Applications

Let $X_n$ be a random walk, and let $\alpha(x) = \mathbb{P}_x \{T_0 < T_N\}$, where $0 \leq x \leq N$. Suppose we are in the case where $p \neq q$. We have seen before that $M_{n \overset{\text{def}}{=} (q/p)^X_n}$ is a fair game. Let $T \overset{\text{def}}{=} T_0 \wedge T_N$ be the first time the walk hits either 0 or $N$. Then $T$ is a stopping time.

Since $M_{T \wedge n}$ is bounded, we can apply Corollary 3.12 to get

$$\mathbb{E}_x \{(q/p)^{X_T}\} = (q/p)^x.$$

We can break up the expectation above to get

$$\mathbb{E}_x \{(q/p)^{X_T}\} = \alpha(x) + (q/p)^N (1 - \alpha(x)).$$
Combining these two equations and solving for $\alpha(x)$ yields

$$
\alpha(x) = \frac{(q/p)^x - (q/p)^N}{1 - (q/p)^N}.
$$

In the case where $p = q = \frac{1}{2}$, we can apply the same argument to get that $\alpha(x) = 1 - (x/N)$.

Now consider again the unbiased random walk. Notice that

$$
\mathbb{E} \left\{ X_{n+1}^2 - X_n^2 \mid X_0, \ldots, X_n \right\} = (X_n + 1)^2 \frac{1}{2} + (X_n - 1)^2 \frac{1}{2} - X_n^2
$$

$$
= 1.
$$

Thus $M_n \overset{\text{def}}{=} S_n^2 - n$ is a fair game. By Theorem 3.11 we have that

$$
\mathbb{E}_x \{ S_{n\wedge T}^2 \} = \mathbb{E}_x \{ T \wedge n \}.
$$

Now since $S_{n\wedge T}^2$ is bounded by $N^2$ for all $n$, if we take the limit as $n \to \infty$ on the left-hand side above, we can take it inside the expectation. Also, $T \wedge n$ does not decrease as $n$ increases, so we are allowed to take the limit inside the expectation. Thus

$$
\mathbb{E}_x \{ S_T^2 \} - x^2 = \mathbb{E}_x \{ T \}.
$$

Now conditioning on whether $T = T_0$ or $T = T_N$ yields

$$
(1 - \alpha(x))N^2 - x^2 = \mathbb{E}_x \{ T \}.
$$

Hence,

$$
\mathbb{E}_x \{ T \} = x(N - x).
$$