Solution to Homework 3.

Problem 1.
Assume the equality holds for \( n \). Then
\[
\mu^{n+1} = \mu P^n P
\]
\[
= \left[ \frac{1}{2} (1 + 2^{-n}) \cdot \frac{1}{2} (1 - 2^{-n}) \right] \left[ \begin{array}{cc} 3/4 & 1/4 \\ 1/4 & 3/4 \end{array} \right]
\]
\[
= \left[ \frac{1}{2} (1 + 2^{-n}) \frac{3}{4} + \frac{1}{2} (1 - 2^{-n}) \frac{1}{4} \frac{1}{2} (1 + 2^{-n}) \frac{1}{4} + \frac{1}{2} (1 - 2^{-n}) \frac{3}{4} \right]
\]
\[
= \frac{1}{8} \left[ 3 + 3 \cdot 2^{-n} + 1 - 2^{-n}, 1 + 2^{-n} + 3 - 3 \cdot 2^{-n} \right]
\]
\[
= \frac{1}{8} \left[ 4 + 2 \cdot 2^{-n}, 4 - 2 \cdot 2^{-n} \right]
\]
\[
= \frac{1}{2} \left[ 1 + 2^{-(n+1)}, 1 - 2^{-(n+1)} \right]
\]
\[
= \left[ \frac{1}{2} (1 + 2^{-(n+1)}), \frac{1}{2} (1 - 2^{-(n+1)}) \right].
\]
Thus it holds for \( n + 1 \). It clearly holds for \( n = 0 \). Thus it holds for all \( n \).
We have
\[
\lim_{n \to \infty} \mu^n = \left[ \frac{1}{2}, \frac{1}{2} \right].
\]

Problem 2. Notice that if \( (Y_{n-1}, Y_n) = (0, 1) \), then we must have that \( X_{n-1} = 1 \), as \( X_{n-1} = 1 \) only if \( Y_{n-1} = 0 \). This implies that \( X_n = 2 \). On the other hand, we have that if \( (Y_{n-1}, Y_n) = (1, 1) \), then neither \( X_{n-1} \) or \( X_n \) can be 1. So, in this case, the possibilities for \( (X_{n-1}, X_n) \) are:
\[
(2, 2), (2, 3), (3, 2), (3, 3).
\]
The only one of these that has positive probability is \( (2, 3) \). Thus, if \( (Y_{n-1}, Y_n) = (1, 1) \), then it must be that \( (X_{n-1}, X_n) = (2, 3) \).
We have
\[
\mathbb{P} \left\{ Y_{n+1} = 1 \mid Y_{n-1} = 1, Y_n = 1 \right\} = \mathbb{P} \left\{ Y_{n+1} = 1 \mid X_{n-1} = 2, X_n = 3 \right\}
\]
\[
= 0,
\]
because given that \( X_n = 3 \), we have with probability one that \( X_{n+1} = 1 \) and hence \( Y_{n+1} = 0 \).

But

\[
\mathbb{P} \{ Y_{n+1} = 1 \mid Y_{n-1} = 0, Y_n = 1 \} = \mathbb{P} \{ Y_{n+1} = 1 \mid X_{n-1} = 1, X_n = 2 \} = 1,
\]

since \( X_{n+1} \) must be 3 when \( X_n = 2 \).

Thus it cannot be that

\[
\mathbb{P} \{ Y_{n+1} = 1 \mid Y_0 = i_0, \ldots, Y_n = i_n \} = \mathbb{P} \{ Y_{n+1} = 1 \mid Y_n = i_n \},
\]
as this would imply

\[
0 = \mathbb{P} \{ Y_{n+1} = 1 \mid Y_{n-1} = 1, Y_n = 1 \} = \mathbb{P} \{ Y_{n+1} = 1 \mid Y_n = 1 \} = \mathbb{P} \{ Y_{n+1} = 1 \mid Y_{n-1} = 0, Y_n = 1 \} = 1.
\]

Problem 3.

Just check that

\[
\mathbb{P} \{ X_{2n+2} = j \mid X_{2n} = i \} = (P^2)_{i,j}.
\]

Problem 4.

Take any state \( j \). Since the chain is irreducible, there exist \( r \) and \( s \) so that \( P^{r}_{i,j} > 0 \) and \( P^{s}_{j,i} > 0 \). We know that

\[
P^{r+s}_{j,i} \geq P^{s}_{j,i} P^{r}_{i,j} > 0.
\]

Also, we know that

\[
P^{r+s+1}_{j,i} \geq P^{s}_{j,i} P^{r}_{i,j} > 0.
\]

Thus the set of integers

\[
A = \{ n : P^{n}_{j,j} > 0 \}
\]
contains \( r + s \) and \( r + s + 1 \). But the g.c.d. of \( r + s \) and \( r + s + 1 \) is 1, and so the g.c.d. of \( A \) is 1. Thus state \( j \) has period 1. Since this holds for any \( j \), the Markov chain is aperiodic.

Problem 5. The king is irreducible, as it can reach any square. It is possible to return to any position in both 2 moves and 3 moves, so it is aperiodic.

The bishop is restricted to its starting color, so it is not irreducible. It is possible to return to its position in both 2 and 3 moves, so it is aperiodic.

The knight is irreducible, but has period two. To see that it has period two, notice that is always moves from a white position to a black position, and from a black position to a white position. Thus it can only return to its starting place after an even number of moves.