Homework M5040 - Levin

Let \( \{X_n\} \) be an unbiased random walk on \( \mathbb{Z} \). The goal of this assignment is to find the distribution of \( T = T_{-1} = \min \{ n : X_n = -1 \} \). Note by symmetry that the distribution of \( T \) is the same as the distribution of \( T_1 \), the first time the walk reaches 1.

We will use generating functions. Let \( A(s) = \mathbb{E} \{ s^T \} \). Notice that by the definition of expectation,

\[
\mathbb{E} \{ s^T \} = \sum_{k=0}^{\infty} \mathbb{P} \{ T = k \} s^k.
\]

Suppose we find \( A(s) \), and can write it as a power series \( A(s) = \sum_k a_k s^k \). If a function has a power series representation (such functions are called real analytic), then the coefficients are unique. Thus \( a_k = \mathbb{P} \{ T = k \} \).

1. Show that \( M_n \overset{\text{def}}{=} u X_n \left( \frac{2}{u + 1/u} \right)^n \) is a fair game.

2. Show that for \( u \) near 0, there is a constant \( K \) so that \( |M_n \wedge T| < K \).

3. Then use the Corollary of the notes to show that

\[
\mathbb{E} \left\{ \left( \frac{2}{u + 1/u} \right)^T \right\} = u.
\]

4. Use (1) to show that

\[
\mathbb{E} \{ s^T \} = \frac{1}{s} \left( 1 - \sqrt{1 - s^2} \right).
\]

5. Show by induction that if \( f(x) = \sqrt{1 - x} \), then

\[
f^{(m)}(x) = \frac{(2m - 2)!}{2^{2m-1} (m-1)!} (1-x)^{-(2m-1)/2}.
\]

6. Use this to show that

\[
1 - \sqrt{1 - s^2} = \sum_{m=1}^{\infty} \frac{(2m - 2)!}{2^{2m-1} m! (m-1)!} s^{2m}.
\]
7. Conclude that

\[ A(s) = \sum_{m=1}^{\infty} \frac{(2m - 2)!}{2^{2m-1} m! (m-1)!} s^{2m-1}. \]

8. Now argue that

\[
\mathbb{P} \{ T = n \} = \begin{cases} 
  \frac{(2m-2)!}{2^{2m-1} m! (m-1)!} & \text{if } n = 2m - 1 \\
  0 & \text{if } m \text{ is even.}
\end{cases}
\]