Name: ________________________________

Instructions. READ CAREFULLY:

(i) The work you turn in must be your own. You may not discuss the midterm with anyone, either in the class or outside the class. [You may of course consult with me for clarification of any of the problems.]  **Failure to follow this policy will be considered cheating and will result in a course grade of E.**

(ii) You may consult the textbook and your notes. You may consult a textbook on calculus. You *may not* use any other written source. **Failure to follow this policy will be considered cheating and will result in a course grade of E.**

(iii) Your midterm must be clearly written and legible. **I will not grade problems which are sloppily presented and such problems will receive a grade of 0.** If you are unable to write legibly and clearly, use of a word processor. Budget time for writing up your solutions.

(iv) Think about your exposition. Someone (me) has to read what you have written. Your answer is only correct if I can understand what you have done. I reserve the right to deduct points for style, grammar, and spelling.

(v) Midterms are due at Wednesday November 24 at 9:40. **Late midterms will not be accepted.**

Sign below to indicate you have read and understand these instructions.

________________________________________________________________________

1
Problem 1. Let $X$ and $Y$ be jointly continuous with the following joint probability density function:

$$f(s, t) = \begin{cases} 
  c(s^3 + t^3) & \text{if } s \geq 0 \text{ and } t \geq 0 \text{ and } s + t \leq 1, \\
  0 & \text{otherwise}.
\end{cases}$$

(a) Determine $c$.

(b) Find the probability density function for the random variable $Z = X + Y$.

Solution. Note: The case $f(s, t) = c|s^3 + t^2|1\{s, t \geq 0, s + t \leq 1\}$ was done in class. The current problem is a small modification of this.

We have

$$
\begin{align*}
\int_0^1 \int_0^{1-s} (s^3 + t^3) \, dt \, ds &= \int_0^1 \left[ s^3 t + \frac{1}{4} t^4 \right]_0^{1-s} \, ds \\
&= \int_0^1 \left[ s^3 (1-s) + \frac{1}{4} (1-s)^4 + \frac{1}{4} (1-s)^4 \right] \, ds \\
&= \int_0^1 \left[ s^3 - s^4 + \frac{1}{4} (1-s)^4 \right] \, ds \\
&= \left[ \frac{1}{4} s^4 - \frac{1}{5} s^5 - \frac{1}{20} (1-s)^5 \right]_0^1 \\
&= \frac{1}{4} \frac{1}{5} \frac{1}{20} \\
&= \frac{1}{10}
\end{align*}
$$

Thus we should take $c = 10$ so that $f$ integrates to 1.

Now for $0 \leq u \leq 1$,

$$F_Z(u) = P\{X + Y \leq u\} = P\{(X, Y) \in A_u\} = \int \int_{A_u} f(s, t) \, dt \, ds,$$
where

\[ A_u = \{(s, t) : s + t \leq 1, s \geq 0, t \geq 0\}. \]

The region \( A_u \) is sketched below:

We integrate:

\[
\iint_{A_u} f(s, t) dt \, ds = \int_0^u \left[ \int_0^{u-s} 10(s^3 + t^3) \, dt \right] \, ds
\]

\[
= 10 \int_0^u \left[ s^3 t + \frac{1}{4} t^4 \right]_0^{u-s} \, ds
\]

\[
= 10 \int_0^u \left[ s^3(u - s) + \frac{1}{4}(u - s)^4 \right] \, ds
\]

\[
= 10 \int_0^u \left[ us^3 - s^4 + \frac{1}{4}(u - s)^4 \right] \, ds
\]

\[
= 10 \left[ us^4 - \frac{s^5}{5} - \frac{(u - s)^5}{20} \right]_0^u
\]

\[
= 10 \left[ \frac{u^5}{4} - \frac{u^5}{5} + \frac{u^5}{20} \right]
\]

\[
= 10u^5 \left[ \frac{1}{4} - \frac{1}{5} + \frac{1}{20} \right]
\]

\[= u^5. \]
This makes sense: we have $F(0) = 0$ and $F(1) = 1$. To get the density, differentiate:

$$f_Z(u) = \begin{cases} 
5u^4 & \text{if } 0 \leq u \leq 1 \\
0 & \text{if } u < 0 \text{ or } u > 1.
\end{cases}$$
Problem 2. Let $N$ be a Poisson random variable with parameter $\lambda$. Suppose that, given $N = n$, that $X$ is a Binomial$(n, p)$ random variable.

(a) (3 points) Find the joint mass function of $X$ and $N$.

(b) (6 points) Find the (unconditional) mass function of $X$.

(c) (6 points) Find the conditional mass function of $N$ given that $X = k$.

(d) (2 points) Find the conditional mass function of $N - k$ given that $X = k$.

(e) (3 points) Suppose that a Poisson($\lambda$) number of balls are dropped into $r$ boxes, so that the balls are independent, and a ball is equally likely to fall into each box. Find the mass function for the number of balls falling into the first box.

Solution. Recall that given a conditional pmf $p_{X|N}$ and a marginal mass function $p_N$, that the joint can be found as the product:

$$p_{X,N}(k, n) = p_{X|N}(k | n) p_N(n).$$

Thus, for $n \geq k$,

$$p_{X,N}(k, n) = P(X = k, N = n)$$

$$= P(X = k | N = n) P(N = n)$$

$$= \binom{n}{k} p^k (1 - p)^{n-k} \frac{e^{-\lambda} \lambda^n}{n!}$$

$$= \frac{e^{-\lambda} p^k (1 - p)^{n-k} \lambda^n}{k!(n-k)!}$$

Thus

$$p_{X,N}(k, n) = \begin{cases} 
\frac{e^{-\lambda} p^k (1 - p)^{n-k} \lambda^n}{k!(n-k)!} & \text{if } k \geq n \\
0 & \text{otherwise}
\end{cases}$$
\[ p_X(k) = \sum_n p_{X,N}(n) \]
\[ = \sum_{n=k}^{\infty} \frac{e^{-\lambda} p^k (1-p)^{n-k} \lambda^n}{k!(n-k)!} \]
\[ = \frac{e^{-\lambda} (p\lambda)^k}{k!} \sum_{n=k}^{\infty} \frac{[\lambda(1-p)]^{n-k}}{(n-k)!} \]
\[ = \frac{e^{-\lambda} (p\lambda)^k}{k!} \sum_{m=0}^{\infty} \frac{[\lambda(1-p)]^m}{m!} \]
\[ = \frac{e^{-\lambda} (p\lambda)^k}{k!} \]

Thus, \( X \) has a Poisson\((\lambda p)\) distribution.

We have for \( n \geq k \),
\[ p_{N|X}(n \mid k) = \frac{p_{X,N}(n, k)}{p_X(k)} \]
\[ = \frac{\frac{e^{-\lambda} p^k (1-p)^{n-k} \lambda^n}{k!(n-k)!}}{\frac{e^{-\lambda} (p\lambda)^k}{k!}} \]
\[ = \frac{e^{-\lambda(1-p)} [\lambda(1-p)]^{n-k}}{(n-k)!} \]

Also,
\[ P[N - k = m \mid X = k] = P[N = m + k \mid X = k] \]
\[ = \frac{e^{-\lambda(1-p)} [\lambda(1-p)]^m}{m!} \]

Thus, given \( X = k \), then \( N - k \) has a Poisson\((\lambda (1-p))\) distribution.

By the previous results, the number of balls falling in the first box is Poisson\((\lambda / r)\). \( \Box \)
Problem 3. Suppose an urn initially has 5 black balls and 5 white balls. A ball is drawn, and its color noted. This ball, together with a new ball of the same color, is placed back in the urn. So, for example, if a black ball is drawn, two black balls are put back in the urn, so that there are now 6 black balls and 5 white balls. After this, another ball is drawn. Let \( N \) be the number of times in these two draws that a black ball is drawn. Find \( E(N) \) and \( V(N) \).

Hint: Write \( N = X_1 + X_2 \), where

\[
X_i = \begin{cases} 
1 & \text{if a black ball is drawn on the } i\text{th draw} \\
0 & \text{otherwise.}
\end{cases}
\]

Solution. We have

\[
E(N) = E(X_1) + E(X_2).
\]

\[
E(X_1) = 1 \times P(X_1 = 1) + 0 \times P(X_1 = 0)
\]

\[
= P(X_1 = 1)
\]

\[
= \frac{5}{10} = \frac{1}{2}
\]

and

\[
E(X_2) = 1 \times P(X_2 = 1) + 0 \times P(X_2 = 0)
\]

\[
= P(X_2 = 1)
\]

\[
= P(X_2 = 1 \mid X_1 = 1)P(X_1 = 1) + P(X_2 = 1 \mid X_1 = 0)P(X_1 = 0)
\]

\[
= \frac{6}{11} \times \frac{5}{10} + \frac{5}{11} \times \frac{5}{10}
\]

\[
= \frac{1}{2}.
\]

Thus,

\[
E(N) = \frac{1}{2} + \frac{1}{2} = 1.
\]

We have

\[
E(N^2) = E(X_1^2) + 2E(X_1X_2) + E(X_2^2).
\]
Notice that since $X_i$ is equal to either 0 or 1, it follows that $X_i^2 = X_i$. Consequently,

$$E[N^2] = \frac{1}{2} + 2P[X_1 = 1, X_2 = 1] + \frac{1}{2}$$

$$= 1 + 2 \cdot \frac{5}{10} \cdot \frac{6}{11}$$

$$= \frac{17}{11}$$

So

$$V(N) = \frac{17}{11} - 1 = \frac{6}{11}.$$
Problem 4. Let $D$ be the region in the plane indicated in the figure below:

![Diagram](image)

The border on the left side is described by the line $t = s$ and the border on the right side is described by the function $t = s - 1$.

Suppose that $X$ and $Y$ are jointly continuous with the following joint pdf:

$$f(s, t) = \begin{cases} 
1 & \text{if } (s, t) \text{ is in the region } D \\
0 & \text{if } (s, t) \text{ is outside the region } D.
\end{cases}$$

(a) Find the **marginal probability density function** for $X$. [This is the density function for the random variable $X$ alone.]

(b) Find the **marginal probability density function** for $Y$.

(c) Are $X$ and $Y$ independent. Justify your answer.

**Solution.** If $0 \leq s < 1$ then

$$f_X(s) = \int_0^s f(s, t) \, dt = \int_0^s 1 \, dt = s.$$ 

If $s \leq 1 \leq 2$ then

$$f_X(s) = \int_{s-1}^1 f(s, t) \, dt = \int_{s-1}^1 1 \, dt = 2 - s.$$
Then

\[ f_X(s) = \begin{cases} 
  s & \text{if } 0 \leq s < 1, \\
  2 - s & \text{if } 1 \leq s \leq 2 \\
  0 & \text{otherwise}. 
\end{cases} \]

Also if \( 0 \leq t \leq 1 \) then

\[ f_Y(t) = \int_{t}^{t+1} f(s, t) \, ds = \int_{t}^{t+1} ds = 1. \]

So

\[ f_Y(t) = \begin{cases} 
  1 & \text{if } 0 \leq t \leq 1, \\
  0 & \text{otherwise}. 
\end{cases} \]

Check that \( f(s, t) \neq f_X(s) f_Y(t) \).
Problem 5. Let $X$ and $Y$ be independent geometric random variables with parameters $p$ and $q$, respectively.

(a) (5 points) Find the distribution function of $Z = \min\{X, Y\}$.

(b) (15 points) Recall that we interpret the Geometric random variables as the waiting time for a success in a series of independent trials. Describe, using this interpretation, how to arrive at your answer without any calculation.

Solution. Note: This is essentially the same calculation as Exercise 6.T23, which was done in class. We showed that the d.f. for $M = \min\{X_1, X_2, \ldots, X_n\}$, where the $X_i$ are independent and have the same d.f. $F$, is $F_M(t) = 1 - [1 - F(t)]^n$.

$$F_Z(t) = P[Z \leq t]$$
$$= P[Z \leq \lfloor t \rfloor]$$
$$= P[\min\{X, Y\} \leq \lfloor t \rfloor]$$
$$= 1 - P[\min\{X, Y\} > \lfloor t \rfloor]$$
$$= 1 - P[X > \lfloor t \rfloor, Y > \lfloor t \rfloor]$$
$$= 1 - P[X > \lfloor t \rfloor]P[Y > \lfloor t \rfloor]$$
$$= 1 - (1 - p)^{\lfloor t \rfloor}(1 - q)^{\lfloor t \rfloor}$$
$$= 1 - [(1 - p)(1 - q)]^{\lfloor t \rfloor}$$

This is then a geometric random variable with parameter

$$1 - (1 - p)(1 - q) = 1 - (1 - p - q + pq) = p + q - pq$$

We can deduce this without any calculation: Suppose we have two sequences of independent trials, and the two sequences are independent. The trials in the first sequence have probability $p$ of success, and those in the second sequence have probability $q$ of success. $X$ is the number of trials in the first sequence until a success, while $Y$ is the number of trials in
the second sequence until a success. Thus $Z$ is the number of trials until there is a success in either the first or second sequence. The probability of a success, in a given trial, of a success in either sequence is $p + q - pq$. Thus $X$ is Geometric with parameter $p + q - pq$. □