Problem 1. Suppose that the pair of random variables \((X, Y)\) has joint probability density function given by

\[
f(x, y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \leq 1, \\ 0 & \text{if } x^2 + y^2 > 1. \end{cases}
\]

(a) Compute \(P(X^2 + Y^2 > \left(\frac{3}{4}\right)^2)\).

(b) Find the marginal probability density function of \(X\).

(c) Prove or disprove: \(X\) and \(Y\) are independent.

Solution.

Recall from our discussion that for any subset \(A\) contained in the disk of radius 1 centered at \((0,0)\),
\[ P((X, Y) \in A) = \int \int_A f(x, y) \, dx \, dy = \int \int_A \frac{1}{\pi} \, dx \, dy = \frac{\text{Area}(A)}{\pi}. \]

In this particular case, \( A \) is the shaded region in the figure. Thus, \( \text{Area}(A) = \pi - \pi(2/3)^2 \).

We conclude that
\[ P(X^2 + Y^2 > (2/3)^2) = \frac{\pi - \pi(2/3)^2}{\pi} = 1 - (2/3)^2. \]

The marginal density is obtained by integrating out \( y \):
\[ f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy. \]

There are two cases to consider: where \( x < -1 \) or \( x > -1 \), in which case \( f_X(x) = 0 \), and where \(-1 \leq x \leq 1\), in which case we have
\[ f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} \, dy = \frac{2\sqrt{1-x^2}}{\pi}. \]

Thus,
\[ f_X(x) = \begin{cases} \frac{2\sqrt{1-x^2}}{\pi} & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1. \end{cases} \]

Similarly,
\[ f_Y(y) = \begin{cases} \frac{2\sqrt{1-y^2}}{\pi} & \text{if } |y| < 1, \\ 0 & \text{if } |y| > 1. \end{cases} \]

We have for \(-1 \leq x, y \leq 1\),
\[ f_X(x) f_Y(y) = \frac{4}{\pi^2} \sqrt{1-x^2} \sqrt{1-y^2} \neq f(x, y), \]
so that \( X \) and \( Y \) are not independent. \( \square \)