Math 1050-1
Spring 2004
Test 3

NAME

Show all your work!
Problem 1. Graph the following function:

\[ f(x) = \frac{x + 4}{x^2 + 3x - 18}. \]

Label all features of the graph.

Solution. The denominator factors as

\[ x^2 + 3x - 18 = (x + 6)(x - 3), \]

and so is zero at \( x = -6 \) and \( x = 3 \). Thus \( f \) has vertical asymptotes at \( x = -6 \) and \( x = 3 \).

The numerator is 0 at \( x = -4 \), and thus the function has a single zero at \( x = -4 \).

Since the numerator has degree one, which is smaller than the degree of the denominator (two), there is a horizontal asymptote at \( y = 0 \).

Now we only need to check the value of the function at a point between each 0 and vertical asymptote:

<table>
<thead>
<tr>
<th>interval</th>
<th>test ( x ) in interval</th>
<th>( f(x) )</th>
<th>±</th>
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</thead>
<tbody>
<tr>
<td>((-\infty, -6))</td>
<td>-7</td>
<td>(-\frac{3}{10})</td>
<td>−</td>
</tr>
<tr>
<td>((-6, -4))</td>
<td>-5</td>
<td>(\frac{5}{2})</td>
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<tr>
<td>((-4, 3))</td>
<td>0</td>
<td>(-\frac{3}{2})</td>
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<tr>
<td>((3, \infty))</td>
<td>4</td>
<td>(\frac{2}{5})</td>
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\[ \square \]
Problem 2. Below is a graph of \( f(x) = x^3 + 3x^2 - x + 12 \). Find all its (possibly complex) roots.

Solution. Because \( f \) has a zero at \( x = -4 \) (seen from the graph), \( x + 4 \) is a factor. Long division gives

\[
\begin{array}{c|ccccc}
   x + 4 & x^2 - x + 3 \\
   \hline
   x^3 & +3x^2 & -x & +12 \\
   x^3 & +4x^2 & -x \\
   \hline
   -x^2 & -4x \\
   \hline
   3x & +12 \\
   3x & +12 \\
   \hline
   0 & 
\end{array}
\]

That is,

\[ f(x) = (x + 4)(x^2 - x + 3) . \]

To find the roots of \( f \), we must find the roots of the quadratic. The quadratic formula gives

\[
x = \frac{1 \pm \sqrt{1 - 4(1)(3)}}{2} \\
= \frac{1 \pm \sqrt{-11}}{2} \\
= \frac{1}{2} \pm i\frac{\sqrt{11}}{2} .
\]
Thus the three roots of $f$ are

$$-4, \quad \frac{1}{2} + i \frac{\sqrt{11}}{2}, \quad \frac{1}{2} - i \frac{\sqrt{11}}{2}.$$
Problem 3. Compute each of the following

(i) $\log_\pi (\pi^\pi)$

(ii) $\sqrt{3^{\log_\pi (3^3)}}$

(iii) $\log_{123}(e) + \log_{123} \left( \frac{123^{3/2}}{e} \right)$.

Solution.

$\log_\pi (\pi^\pi) = \pi \log_\pi (\pi) = \pi$.

$\sqrt{3^{\log_\pi (3^3)}} = 3^3 = 9$.

$\log_{123}(e) + \log_{123} \left( \frac{123^{3/2}}{e} \right) = \log_{123} \left( e \cdot \frac{123^{3/2}}{e} \right) = \log_{123} 123^{3/2} = \frac{3}{2} \log_{123} 123 = \frac{3}{2}$
Problem 4. Suppose that a stock price is cut in half every 10 days. It starts out at 45 dollars/share. Let \( f(t) \) be the value of the stock after \( t \) days.

(i) Find \( f(t) \).

(ii) After how many days will the stock be worth 1 dollar/share?

Solution.

\[
f(t) = 45 \left( \frac{1}{2} \right)^{t/10},
\]
either by remembering the formula, or by reasoning as follows:

\( f(t) \) must be an exponential function, so \( f(t) = Ba^t \) for some \( B \) and some \( a \). We know

\[
f(t + 10) = \frac{1}{2} f(t),
\]
so plugging in the formula \( f(t) = Ba^t \) gives

\[
\frac{1}{2} = \frac{Ba^{t+10}}{Ba^t} = a^{10}.
\]

Solving for \( a \) gives \( a = \left( \frac{1}{2} \right)^{1/10} \). We know that \( B \) is the initial value, so \( B = 45 \).

For the second part, we need to solve the equation \( f(t) = 1 \) for \( t \).

\[
1 = (45)2^{-t/10}
\]
\[
\frac{1}{45} = 2^{-t/10}
\]

\[
\log_2 \left( \frac{1}{45} \right) = -\frac{t}{10}
\]

\[
-10\log_2 \left( \frac{1}{45} \right) = t.
\]

Thus after \( t = -10\log_2 \left( \frac{1}{45} \right) \) days, the stock is worth 1 dollar/share. To compute the \( \log_2 \), use the change of base formula:

\[
\log_2 \left( \frac{1}{45} \right) = \frac{\ln \left( \frac{1}{45} \right)}{\ln 2}.
\]

The answer simplifies to 54.92.
Problem 5. Solve the following equation for $x$:

$$\log_4(3x^3 - 5) = 3.$$ 

Solution.

\[
\begin{align*}
\log_4(3x^3 - 5) &= 3 \\
4^{\log_4(3x^3 - 5)} &= 4^3 \\
3x^3 - 5 &= 64 \\
3x^3 &= 69 \\
x^3 &= 23 \\
x &= 23^{1/3}.
\end{align*}
\]
Problem 6. Solve for $x$:

\[ e^{\sqrt{x} - 2} = \sqrt{3}. \]

Solution.

\[
e^{\sqrt{x} - 2} = \sqrt{3} \\
\ln \left( e^{\sqrt{x} - 2} \right) = \ln(\sqrt{3}) \\
\sqrt{x} - 2 = \ln(\sqrt{3}) \\
\sqrt{x} = 2 + \ln(\sqrt{3}) \\
x = \left( 2 + \ln(\sqrt{3}) \right)^2.
\]
Problem 7. Suppose a function $f$ of $x$ triples every positive change of 5 units in $x$, and has initial value 6. What is $f(x)$?

Solution. One either remembers the formula, and gets

$$f(x) = (6)3^{x/5},$$

or reasons as follows:

The function has the form $f(x) = Ba^x$, since it is of exponential form. We know $B$ must be the initial value 6. We have $f(x + 5) = 3f(x)$, so plugging in the formula $f(x) = Ba^x$ gives

$$Ba^{x+5} = 3Ba^x$$
$$a^5 = 3$$
$$a = 3^{1/5}.$$

Thus $f(x) = (6)(3^{1/5})^x = (6)3^{x/5}$.
Problem 8. Suppose you deposit 250 dollars in a bank account which compounds interest continuously. After 140 days, you have 270 dollars. What is the nominal interest rate?

Solution. We have \( f(t) = Pe^{rt} \), where \( P \) is the initial investment, and \( r \) is the nominal interest rate. \( P = 250 \) here.

\[
\frac{270}{250} = \frac{f(140)}{250} = \frac{250e^{r140}}{250} = e^{140r}.
\]

This gives the equation \( 27/25 = e^{140r} \). Solving for \( r \) gives:

\[
e^{140r} = \frac{27}{25} \\
140r = \ln \left( \frac{27}{25} \right) \\
r = \frac{1}{140} \ln \left( \frac{27}{25} \right) \\
r = 0.000549722.
\]

This gives the interest rate in units of days.
Problem 9. Suppose a population grows at rate 100 individuals per day from an initial size of 20000. The population size approaches a stable maximum size of 235000. Find the function giving population size as a function of time (measured in days).

Solution. Because the population size stabilizes, we use a logistic function:

\[ f(x) = \frac{a}{1 + be^{-rt}}. \]

Here \( r = 100 \), and \( a = 235000 \). The only thing left is to determine \( b \). We use the equation \( f(0) = 20000 \) to find \( b \):

\[
20000 = f(0) = \frac{235000}{1 + b} \\
20000(1 + b) = 235000 \\
1 + b = \frac{235000}{20000} \\
b = \frac{235}{20} - 1 \\
b = \frac{43}{4}.
\]

Thus

\[ f(t) = \frac{235000}{1 + \frac{43}{4}e^{-100t}}. \]
The decimal point is 1 digit(s) to the right of the |

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Min. 1st Qu. Median Mean 3rd Qu. Max.
6.00 44.00 65.00 61.33 81.00 90.00