PART III: REVOLUTION

How should one think of fractal construction in general?
Is it really "deletion"?
(Cantor, Koch, etc. were "wrong"!)

This new way of thinking is due to
John Hutchinson

"Fractals and self-similarity"
Indiana University Math Journal
30 (1981), 713-747
John Hutchinson "Fractals and self-similarity"
Indiana University Math Journal 30 (1981), 713-747
Let $X$ be a space on which you can measure distance

\[
\begin{align*}
\text{dist}(P, Q) &\geq 0 \quad (\text{only when } P = Q) \\
\text{dist}(P, Q) &= \text{dist}(Q, P) \\
\text{dist}(P, R) &\leq \text{dist}(P, Q) + \text{dist}(Q, R)
\end{align*}
\]

$X$ is called a metric space.

A transformation $f$ of $X$ is a contraction if there is a fraction $\mu$, $0 < \mu < 1$ so that

\[
\text{dist}(f(P), f(Q)) \leq \mu \text{dist}(P, Q)
\]

Example $f: [-10, 10] \to \mathbb{R}$

$f(x) = \frac{1}{2}x$

Is $f$ a contraction?

Which $x$'s satisfy $f(x) = x$? These are called fixed points.

If we pick any $x_0$ and set $x_1 = f(x_0)$, $x_2 = f(x_1)$, $x_3 = f(x_2)$, what happens to $x_n$ as $n \to \infty$?
Example: \( A : L\text{-box} \rightarrow L\text{-box} \)

\[
\mathcal{N} \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x + \frac{1}{4} \\ \frac{1}{2}y + \frac{1}{4} \end{bmatrix}
\]

Is \( A \) a contraction?

Does it have a fixed point?

If we start anywhere, and iterate, do we get to the fixed point?
Banach Fixed Point Principle
(Contracted Mapping Theorem)

- $X$ a complete metric space
- $f : X \rightarrow X$ a contraction mapping

**Conclusion**: $f$ has a unique fixed point $P$ in $X$, i.e. $f(P) = P$

And for any $z_0$ in $X$ the iteration

$z_{n+1} = f(z_n)$, $n = 0, 1, 2, \ldots$

yields a sequence so that $d(z_n, P) \rightarrow 0$ as $n \rightarrow \infty$
Stefan Banach

Born: 30 March 1892 in Kraków, Austria-Hungary (now Poland)
Died: 31 Aug 1945 in Lvov, Ukraine
Ingredient 2: Hausdorff distance

Bob, fractals, are all (closed and bounded) subsets of $\mathbb{R}^2$

Hausdorff figured out a good way to measure distance on the space of these subsets

\[ \text{dist}(A, B) = \text{the larger of } \left\{ \begin{array}{l} \text{the maximum distance from points in } A \\ \text{to (the nearest point) in } B \end{array} \right\} \quad \text{or} \quad \left( \begin{array}{l} \text{the maximum distance from points in } B \\ \text{to (their nearest point) in } A \end{array} \right) \]
Felix Hausdorff

Born: 8 Nov 1869 in Breslau, Germany (now Wroclaw, Poland)
Died: 26 Jan 1942 in Bonn, Germany
Let $S$ be a set

$$F(S) := f_1(S) \cup f_2(S) \cup f_3(S)$$

- $F$ is a contraction, using Hausdorff distance.
- Sierpinski $\Delta$ is its fixed point!
S  F(S)  F(F(S))  F(F(F(S))) ...

Figure 5.1: Three iterations of an MRCM with three different initial images.

If we started with Bob he would turn into Sierpinski!
Hutchinson's Theorem 1981

- $X$ a complete metric space

- $f_i : X \to X$ contractions with contraction constants $k_i$

Let $F : \text{the space of subsets of } X \to \text{the space of subsets of } X$

- $(\text{the "Bo}b\text{"})$

- a metric space with the Hausdorff distance

where $F(\text{Bo}b) := f_1(\text{Bo}b) \cup f_2(\text{Bo}b) \cup \ldots \cup f_n(\text{Bo}b)$

Then $F$ is a contraction, with contraction constant $k = \max (k_1, k_2, \ldots, k_n)$

- So $F$ has a unique fixed point (our fractal!) and we can get it by picking any initial subset and iterating $F$!

[First identify desired fractal & get a fixed point!]
Figure 5.9: IFS with three similarity transformations with scaling factor 1/2.

Figure 5.10: Another IFS with three similarity transformations with scaling factor 1/2.

Figure 5.11: The white line is inserted only to show that the figure can be made up from three parts similar to the whole.

Figure 5.12: IFS with three transformations, one of which is a similarity. The attractor is related to the Cantor set.
IFS for a Twig

Figure 5.13: IFS with three affine transformations (no similarities).

Crystal with Four Transformations

Figure 5.14: IFS with four similarity transformations.

Triangle, Square, and Circle

Figure 5.17: The encoding of a triangle, a square and a circle by IFSs.
Figure 5.15: IFS with five similarity transformations. Can you see Koch curves in the attractor?

A Tree

Leaf Collage