Ingredient 1: Contraction

Let $X$ be a space on which you can measure distance:

\[
\begin{align*}
\text{dist}(P, Q) &> 0; \quad = 0 \text{ only when } P = Q \\
\text{dist}(P, Q) &= \text{dist}(Q, P) \\
d(P, R) &\leq d(P, Q) + d(Q, R)
\end{align*}
\]

$X$ is called a metric space.

A transformation $f$ of $X$ is a contraction if there is a fraction $\mu$, $0 < \mu < 1$ so that

\[
d(f(P), f(Q)) \leq \mu d(P, Q)
\]

Example: $f: [-10, 10] \to \mathbb{R}$

$f(x) = \frac{1}{2} x$

Is $f$ a contraction?

Which $x$'s satisfy $f(x) = x$? These are called fixed points.

If we pick any $x_0$ and set $x_1 = f(x_0)$, $x_2 = f(x_1)$, $x_3 = f(x_2)$, $\ldots$ What happens to $x_n$ as $n \to \infty$?