Exercise 1. Prove that the $p$-norm $\|f\|_p$ satisfies the triangle inequality, $1 < p < \infty$.

*Note:* Use function notation, since this includes the case of $\mathbb{R}^n$ if you identify $\mathbb{R}^n$ with the integrable bounded functions on $[0,1]$.

$$f(x) = \begin{cases} x_1, & 0 \leq t < 1 \\ x_n, & n-1 \leq t < n \end{cases}$$

If $p = \infty$, then

$$\|f\|_\infty = \|f\|_p = \|f\|_p.$$ 

(a) Prove the generalized geometric-arithmetic mean inequality:

$$\alpha, \beta > 0 \Rightarrow \alpha^\beta \leq \alpha + (1-\alpha)\beta \quad \text{(equality iff } \alpha = \beta)$$

*Hint:* Consider, for $t > 0$, $\varphi(t) = (1-\lambda) \lambda t - t^\lambda$.

Use Calculus ($\varphi'(t)$) to show that for $t > 0$, $\varphi$ has a maximum value of 0, at $t = 1$.

Then, for $\beta > 0$, manipulate $\varphi'(t) \leq 0$ to deduce result. Check $\beta = 0$ separately.

(b) Prove the generalized Cauchy-Schwarz inequality (called the Hölder inequality)

$$\int_a^b |f(t)g(t)| \, dt \leq \|f\|_p \|g\|_q$$

where $\frac{1}{p} + \frac{1}{q} = 1 \quad \text{(i.e. } q = \frac{p}{p-1})$

*Hint:* First deduce Hölder in case $\|f\|_p = \|g\|_q = 1$ by deriving $|f(t)g(t)| \leq \|f\|_p \|g\|_q$, from (a), and integrating.

Derive the general case by considering $\|f\|_p$, $\|g\|_q$, and applying what you just derived.

(c) Finally, prove $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ by starting with

$$\|f + g\|_p^p = \int_a^b |f(t) + g(t)|^p \, dt = \int_a^b |f(t) + g(t)|^p \, dt \leq \int_a^b |f(t)|^p + |g(t)|^p \, dt$$

and Höldering these last two inequalities in the only way that would make sense.
Exercise 2
Define \( l^2 = \{ \{ x_n \}_{n \in \mathbb{N}} \} \) s.t. \( \sum_{k=1}^{\infty} x_k^2 < \infty \)

(This is kind of a limit of \( (\mathbb{R}^n, || \cdot ||_{l^2}) \) as \( n \to \infty \)).

2a) Prove that \( \langle v, w \rangle := \sum_{k=1}^{\infty} v_k w_k \) is a well-defined inner product on \( l^2 \)

\( v = \{v_n\}, w = \{w_n\} \)

2b) Prove \( l^2 \) is a Hilbert space
(i.e. it's complete w.r.t the norm \( || \cdot || = \langle \cdot , \cdot \rangle^{1/2} \))

Exercise 3
\( C[0,1] := \{ f : C[0,1] \to \mathbb{R} \text{ s.t. } f \text{ continuous} \} \)

Prove \( (C[0,1], || \cdot ||_p) \) is not complete for any \( 1 \leq p < \infty \)

Exercise 4
Let \((X, d)\) be a metric space.

Define a relation \( \equiv \) on Cauchy sequences \( \{x_n\} \subset X \)

\( \{x_n\} \equiv \{y_n\} \iff \forall \varepsilon \in \mathbb{R} \exists N \in \mathbb{N} \text{ s.t. } k, m > N \Rightarrow d(x_k, y_m) < \varepsilon \)

4a) Show this is an equivalence relation

4b) Define \( \tilde{X} = \) the equivalence classes of Cauchy sequences

Define \( \tilde{d}([\{x_n\}], [\{y_n\}]) = \lim_{k \to \infty} d(x_k, y_k) \)

Show \( \tilde{d} \) exists and is well-defined

4c) Show \((\tilde{X}, \tilde{d})\) is a complete metric space, and that there is a natural "inclusion" map \( \tilde{i} : X \to \tilde{X} \)
\( \tilde{i} : d(xy) = \tilde{d}(i(x), i(y)) \forall x,y \in X \)

Exercise 5
Let \( A \) be a subset of the metric space \((X, d)\)

Write \( \overset{\circ}{A} \) for interior of \( A \)
\( \overline{A} \) for closure of \( A \)
\( A^c \) for complement of \( A \).

Using these three operations, what is the maximum \# of distinct sets you can write, starting with \( A \)?
Find a set for which this max \# occurs, (Hint: You can find one) in \( \mathbb{R} \), with the usual metric.