2.1.22 \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \]

We try for a ruling from the base curve
\[ \beta(u) = (a \cos u, b \sin u, 0), \]
the waist of the hyperboloid.
We expect the line direction to have horizontal component \( \parallel \) to \( \beta'(u) \).
In fact, book says to take
\[ \delta(u) = \beta'(u) + (0, 0, c). \]

Let's try it:
\[ X(u, v) = \beta(u) + v \delta(u) \]
\[ = (a \cos u, b \sin u, 0) + v(-a \sin u, b \cos u, c) \]
\[ = \left( \frac{a^2 \cos^2 u - 2c \cos u \sin u + c^2 \sin^2 u}{a^2}, \frac{a^2 \sin^2 u + 2c \cos u \sin u + c^2 \sin^2 u}{b^2}, 0 \right) \]
\[ = \frac{\cos^2 u - 2c \cos u \sin u + c^2 \sin^2 u}{a^2}, \frac{a^2 \sin^2 u + 2c \cos u \sin u + c^2 \sin^2 u}{b^2}, 0 \]
\[ = 1, \]
\[ = 1. \]

In fact this is a double ruling because we could take
\[ \delta(u) = \beta'(u) + (0, 0, -c) \] as well - the computation above would be virtually identical.

2.7 \[ X_u(s) = \frac{\partial}{\partial u} X(s) \]
\[ = \sum_i \frac{\partial}{\partial u} X_i(s) \]
\[ \therefore X_v(X_u(s)) = X_v \left( \sum_i \frac{\partial}{\partial u} X_i(s) \right) \]
\[ = \sum_i \frac{\partial}{\partial u} X_i(s) \frac{\partial}{\partial v} X_i(s) + \sum_i \frac{\partial}{\partial v} X_i(s) \frac{\partial}{\partial u} X_i(s) \]
\[ = \sum_i \frac{\partial}{\partial u} X_i(s) \frac{\partial}{\partial v} X_i(s) + \sum_i \frac{\partial}{\partial v} X_i(s) \frac{\partial}{\partial u} X_i(s) \]

Notice \[ X_v(X_u(s)) = X_u(X_v(s)), \]
and this must be true, since one equals \((f \circ X)_w\), the other \((f \circ X)_v\).

2.9 \[ Z = \sum_i Z_i \]
\[ \nabla_i Z = \sum_i \nabla_i Z_i = \sum \nabla_i (Z_i) = \sum \nabla_i (\nabla_i Z) = \sum \nabla_i (\nabla_i Z) \]
\[ \therefore \nabla X_v (\nabla_i Z) = \sum_i X_v \left( \frac{\partial^2 Z}{\partial x_i \partial x_j} \right) \frac{\partial}{\partial v} \]
\[ \therefore \nabla_i \nabla Z = \nabla_i \nabla Z \]

Notice \[ \nabla_i \nabla Z = \nabla_i \nabla Z, \] as it must since one equals \( \nabla_i Z \), the other \( \nabla_i Z \).
\( \beta = X^{-1} \alpha \) (where it makes sense).

We must show that if \( \alpha \) is diffeable in a point in \( \mathbb{R}^3 \), as a map to \( \mathbb{R}^3 \) (which happens to lie on \( M \)),
then \( \beta = X^{-1} \alpha \) is too.

**Proof:**

Near \((p(t_0), 0)\) define \( Z(u, v, w) = X(u, v) + w \left( \frac{x_u x_v}{\|x_u x_v\|} \right) \)

(1)

So \( Z(u, v, 0) \) is original \( X(u, v) \)

The derivative matrix \( \frac{\partial Z}{\partial (u, v)} \) at \((u, v, 0) = (q, 0)\) is

\[
\begin{bmatrix}
X_u & X_v & U
\end{bmatrix}
\]

where \( U = \frac{x_u x_v}{\|x_u x_v\|} \), at \( q \).

This matrix is nonsingular at \((q, 0)\) because \( X_u, X_v, U \) are lin. indep.

\[
\begin{bmatrix}
c_1 X_u + c_2 X_v + c_3 U
\end{bmatrix} = 0
\]

then \( c_1 = c_2 = 0 \) since \( X_u, X_v, U \) lin. indep.

Thus, inverse fun then implies

- \( Z \) has a local inverse fun (which is \( C^\infty \)-diffeable since \( Z(0) \)),
- mapping a neighborhood of \( p = X(q) \) back to a neighborhood of \((q, 0)\).

But \( X \circ \beta = \alpha \)

\( \Rightarrow \)

\( Z \circ \beta = \alpha \)

\( \Rightarrow \)

\( \beta = Z^{-1} \alpha \) is composition of diffeable is diffeable.
Show \( Y_0 \cdot X : X^{-1}(Y(D_0)) \to Y^{-1}(X(D_1)) \) is differentiable.

Same notation: trick: note, differentiability is just a local property.

The problem here is \( Y^{-1} \), since \( X \) is differentiable.

So let \( p \in X(D_1) \cap Y(D_0) \)

\[
q_1 = X^{-1}(p) \\
q_2 = Y^{-1}(p)
\]

Near \( (q_2, 0) \) define

\[
Z(r, s, w) = Y(r, s) + w \left( \frac{Y_r \times Y_s}{||Y_r \times Y_s||} \right)
\]

Derivative matrix \( \mathbf{Z} \) at \((q_2, 0)\) is

\[
\begin{bmatrix}
Y_r & Y_s \\
0 & 0
\end{bmatrix}
\]

All entries evaluated at \((u, v) = q_2\).

Since matrix is nonsingular at \((q_2, 0)\), \( Z \) has local inverse, so near \((q_2, 0) \) and \( p^* \) and \( q_2 \),

\[ Y^{-1} \cdot X = Z^{-1} \cdot X \]

is a composition of differentiable maps is differentiable.

Merge patches for surface = revolution

\[
\beta(u) = \langle f(u), g(u) \rangle \\
\beta \text{ regular, } 0 \leq u \leq 1.
\]

Yields \( X(u, v) = \langle f(u), g(u) \cos v, g(u) \sin v \rangle \)

\( 0 \leq v \leq 2\pi \), e.g. (any mer- \( u = \theta \)
midaly length \( 2\pi \) would work)

\( x \cdot 1-1 : X(u, v) = X(u, \tilde{v}) \Rightarrow f(u) = f(\tilde{u}) \) \( (x-\text{cond}) \)

\( x \) is regular:

\( x \) is differentiable.

\( x \) is regular:

\( X_u = \langle f'(u), g'(u) \cos v, g'(u) \sin v \rangle \)

\( X_v = \langle 0, g'(u) \sin v, g'(u) \cos v \rangle \)

\( \langle f, g \rangle \) regular \( \Rightarrow \langle f', g' \rangle \neq 0 \)

\( X_u \neq 0 \)

\( g > 0 \Rightarrow X_v \neq 0 \)

\( X_u, X_v \) linear ill.