Chapter 2: Complex integration
- Leads to Cauchy Integral formula and magic theorems:
  - Liouville: bounded entire functions are constant
  - F.T. Algebra: every degree n polynomial has n (complex) roots (counting multiplicity)
  - Magic ways to calculate integrals (contour integration)

Definition A: For \( f: [a,b] \rightarrow \mathbb{C} \) continuous, \( f(t) = u(t) + iv(t) \),

\[
\text{For} \quad \int_{a}^{b} f(t) \, dt = \int_{a}^{b} u(t) \, dt + i \int_{a}^{b} v(t) \, dt
\]
then
\[
\text{(A1)} \quad \int_{a}^{b} f(t) \, dt = \int_{a}^{b} u(t) \, dt + i \int_{a}^{b} v(t) \, dt
\]
is equivalent to
\[
\text{(A2)} \quad \int_{a}^{b} f(t) \, dt = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j^*) \Delta t_j
\]
where \( P: a = t_0 < t_1 < \ldots < t_n = b \),

\( t_j^* \leq t_j; \Delta t_j = t_j - t_{j-1} \)

Exercise 1

\[
\int_{0}^{\pi/2} -2 \sin t \cos t + i (\cos^2 t - \sin^2 t) \, dt =
\]

A3) Fundamental Theorem of Calculus (apply real FTC to real & imag parts)

\[
\int_{a}^{b} f(t) \, dt = F(b) - F(a) \text{ if } F'(t) = f(t) \text{ V t} \in [a,b].
\]

A4) Integral estimate

\[
\left| \int_{a}^{b} f(t) \, dt \right| \leq \int_{a}^{b} |f(t)| \, dt
\]

pf: apply A2:

\[
\left| \int_{a}^{b} f(t) \, dt \right| = \lim_{\|P\| \to 0} \left| \sum_{j=1}^{n} f(t_j^*) \Delta t_j \right|
\]

\[
= \lim_{\|P\| \to 0} \left| \sum_{j=1}^{n} |f(t_j^*)| \Delta t_j \right|
\]

\[
= \int_{a}^{b} |f(t)| \, dt
\]

HW for Fri 9/23

2.1 2ac, 3, 5, 10, 11, 13, 14

2.2 1 ad, 2 (prove with FTC), 3, 4, 6, 8, 11.

Class exercise, page 4 & today's notes
(1.6 # 6; 10.14 are carried over from last week)

\( i = \Re f \) \( \forall \text{ Im } f \)
Definition 3 Let $A \subset \mathbb{C}$ open, $f : A \to \mathbb{C}$ continuous (not necessarily analytic, although label on this will be our primary focus)

$$\int_{\gamma} f(z) \, dz = \int_{a}^{b} f(\gamma(t)) \gamma'(t) \, dt \quad \text{using Def A!}$$

(2)

(2.1) $\int_{\gamma} f(z) \, dz \quad \gamma$

(so, you substitute $z = \gamma(t)$ and the differential $dz = \gamma'(t) \, dt$ to evaluate the contour integral)

in case $|\gamma'(t)| > 0$ on $[a, b]$ (so $|\gamma'(t)| > \delta > 0$), then (2.1) is equivalent to

$$\int_{\gamma} f(z) \, dz = \lim_{\max_{i} \Delta z_{i} \to 0} \sum_{i} f(z_{i}) \Delta z_{i}$$

$$= \lim_{\max_{i} \Delta t_{i} \to 0} \sum_{i} f(\gamma(t_{i-1})) \gamma'(t_{i}) \Delta t_{i}$$

$$= \int_{a}^{b} f(\gamma(t)) \gamma'(t) \, dt + O$$

$$= \int_{a}^{b} f(\gamma(t)) \gamma'(t) \, dt + O$$

Exercise 2 Let $\gamma(t) = e^{it} \quad 0 \leq t \leq \pi/2$

$f(z) = z$

compute $\int_{\gamma} f(z) \, dz$

Exercise 3 $\gamma(t) = e^{it} \quad 0 \leq t \leq 2\pi$

$f(z) = \frac{1}{z}$

compute $\int_{\gamma} f(z) \, dz$
(B3) FTC: If \( \exists F : A \to \mathbb{C} \) analytic, with \( F' = f \) on \( A \), then
\[
\int_\gamma f(z) \, dz = F(\gamma(b)) - F(\gamma(a))
\]
\[
\text{pf: } \int_\gamma f(z) \, dz = \int_a^b f(\gamma(t)) \gamma'(t) \, dt
\]
\[
= \int_a^b F'(\gamma(t)) \gamma'(t) \, dt = \left[ F(\gamma(t)) \right]^b_a \quad \text{by (A3)}
\]

(B4) Integral Estimate:
\[
\left| \int_\gamma f(z) \, dz \right| \leq \int_a^b |f(\gamma(t))\gamma'(t)| \, dt
\]
\[
\leq \int_a^b |f(\gamma(t))| |\gamma'(t)| \, dt \quad (A4)
\]
\[
= \int_a^b |f(\gamma(t))| |\gamma'(t)| \, dt
\]

(B5): \[
\int_\gamma |f(z)| \, |dz| = \int_a^b |f(\gamma(t))| |\gamma'(t)| \, dt
\]

so (B4) reads \[
\left| \int_\gamma f(z) \, dz \right| \leq \int_\gamma |f(z)| \, |dz|
\]

Exercise 4: Rework Exercise 2 using FTC.

Exercise 5: What is \( \int_\gamma |f(z)| \, |dz| \) for Exercise 2 Example?

Exercise 6: \( \alpha(t) = e^{it}, \ 0 \leq t \leq 2\pi \)
\[ f(z) = z^n \quad n \in \mathbb{Z}. \]
What is \( \int_\gamma f(z) \, dz \)?
Relation of contour integrals to (real) line integrals:

while \( f(z) = u(x, y) + iv(x, y) \)
\[ \gamma(t) = x(t) + iy(t) \]

then

\[
\int_{\gamma} f(z) \, dz = \int_{a}^{b} \left[ u(x(t), y(t)) + iv(x(t), y(t)) \right] \left[ x'(t) + iy'(t) \right] \, dt
\]

\[
= \int_{a}^{b} u(x(t), y(t)) \, x'(t) - v(x(t), y(t)) \, y'(t) \, dt
\]

\[
+ i \int_{a}^{b} v(x(t), y(t)) \, x'(t) + u(x(t), y(t)) \, y'(t) \, dt
\]

\[
= \int_{\gamma} u \, dx - v \, dy + i \int_{\gamma} v \, dx + u \, dy
\]

Easy to derive formally by writing

\( f = u + iv, \quad dz = dx + idy \)

\[
\int_{\gamma} (u + iv)(dx + idy) = \int_{\gamma} u \, dx - v \, dy + i \int_{\gamma} v \, dx + u \, dy.
\]

Class exercise

Change of variables in complex curve integrals: let \( \gamma: [a, b] \to C \subset \mathbb{C} \)
\( \tilde{\gamma}: [c, d] \to \mathbb{C} \)
have the same image curve so that the following diagram commutes, and all functions are \( C^1 \) with non-zero derivatives

Use chain rule to prove

\[
\int_{\tilde{\gamma}} f(z) \, dz = \begin{cases} 
+ \int_{\gamma} f(z) \, dz & \text{if } k(c) = a \text{ and } k(d) = b \\
- \int_{\gamma} f(z) \, dz & \text{if } k(c) = b \text{ and } k(d) = a.
\end{cases}
\]