\[ f(z) = \sqrt{z^2 - 1} = \sqrt{z - 1} \sqrt{z - e^{2\pi i}} \sqrt{z - e^{-2\pi i}} \]

So could do something like

\[ \theta_1 = \text{ang}(z - 1) \]
\[ \theta_2 = \text{ang}(z - e^{2\pi i}) \]
\[ \theta_3 = \text{ang}(z - e^{-2\pi i}) \]

alternately, you could try the composition trick: Pick a
branch of \( \sqrt{w} \) and then see if this branch lets you work back to a
good domain for \( f \). (\( \mathbb{C} \setminus \text{finite rays} \) and \( \text{rays} \), still connected).

If you happen upon

\[ g(z) = z^3 - 1 \]
\[ z^3 \in [1, \infty) \]
\[ z^3 \in (0, \infty) \]

\[ \text{iff } z = r e^{i\theta} \]
\[ r > 1 \]
\[ \theta = 0, 2\pi/3, -2\pi/3 \]

thus inverse image of branch cut is exactly one \( \sqrt[3]{\text{th}} \) preimage!

\[ f(z) = \sqrt{g(z)} \]  
\[ f'(z) = \frac{1}{2} (z^2 - 1)^{-1/2} \]

b) any branch of \( \sqrt{z} \) works, because \( \sin(w) \) is entire.

\[ \sin(z) = \sin(z^2) \]

\[ \sin(z^2) = (\cos(z^2) - z^2) \]
14. \( f(z) = \sqrt{1 + \sqrt{z}} \)

\[ z = |r| e^{i \theta} \]

\[ r = \sqrt{1 + \sqrt{1 + |z|}} \]

\[ \theta = \arg z \]

\[ -\pi < \theta < \pi \]

\[ f'(z) = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{z}} \right) \frac{1}{2} e^{-i \theta} \]

where we make the same choices of argument.
2.1 2a) \[ \int \frac{x}{y} \, dz \]

\[ y_1 \]

\[ y_2 \]

\[ y_1 \]

\[ y_2 \]

\[ z = xy \]

\[ 0 \leq y \leq 1 \]

\[ \int_0^1 (1) \, dy = 0 \]

\[ \int x \, dz = 4 - \] \[ \int_0^2 x \, \frac{y^2}{2} \, dy = 4 \]

So \( \int x \, dz = 4 \)

2c) \[ \int \frac{1}{z-1} \, dz \]

\[ \gamma = \text{circle of radius } 2, \text{ centered at } 0, \text{ c.c.} \]

\[ \gamma(t) = 1 + 2e^{it}, \quad 0 \leq t \leq 2\pi \]

\[ \gamma'(t) = 2ie^{it} \]

\[ \int_0^{2\pi} \frac{1}{2e^{it}} \, 2ie^{it} \, dt = \pi \]

3. \[ \int \frac{1}{z} \, dz \]

\[ \gamma = \text{circle of radius } 1, \text{ centered at } 2, \text{ c.c.} \]

\[ \log z = \ln |z| + i \arg z \quad -\pi < \arg z < \pi \]

which is an open region containing \( \gamma \)

Since \( (\log z)' = \frac{1}{z} \)

and since \( \gamma \) is closed,

you can do this by parameterizing \( \gamma \),

but it is very painful.

\[ \int_\gamma \frac{1}{z} \, dz = 0. \]

5. \[ \oint \gamma (z^2 - 1) \, dz = \oint \gamma \, (dx + i\, dy) \]

\[ \oint \gamma \, (udx - vdy + i \, vdx +udy) \]

\[ \nexists \]

So \( \Re(\oint f(z) \, dz) = \oint \gamma (udx + vdy) \)

\[ \nexists \]

these are almost never equal.

(Counterexamples easy, i.e., \( \Re(\gamma) = \cos t \), \( \Re(\gamma) = \sin t \))

On the other hand, \( \oint \Re f(z) \, dz = \oint \Re \, (dx + i\, dy) = \oint \Re \, (udx + i\, vdy) \)

then \( \Re(\oint f(z) \, dz) = \oint \Re \, (udx + vdy) \)

\[ \Re(\gamma) = \cos t \]

\[ 0 \leq t \leq 1 \]

\[ u = 0 \]

\[ v = 1 \]

\[ \Re(\gamma) = \sin t \]

\[ \Re(\gamma) = \sin t \]

\[ \Re(\gamma) = \sin t \]
11. a) \[ \int \frac{dz}{z} = \int \frac{1}{e^{it}} \cdot e^{i2t} \, dt = \int_0^{2\pi} e^{i2t} \, dt = 2\pi i \]

\[2 = e^{i2}\]
\[\frac{dz}{z} = e^{i2} \, dt\]

\[\text{use this for all of 11a) if necessary}\]

\[\int \frac{dz}{z_1} = \int_0^{2\pi} e^{i2} \, dt = \frac{2\pi}{0} = 0\]

\[\int \frac{1 |dz|}{z_1} = \int_0^{2\pi} |e^{i2}| \, dt = \int_0^{2\pi} e^{-i2} \, dt = -e^{-i2}\]

\[\int \frac{|dz|}{z_1} = \text{Circum} = 2\pi\]

b) \[\int_0^{2\pi} z^2 \, dz = \int_0^{2\pi} \left( \frac{2\pi}{3} \right) e^{i2} \, dt = \frac{2\pi}{3} e^{i2} \int_0^{2\pi} = -\frac{1}{3}\]

13. \[\int_0^{2\pi} 2 \sin^2 \theta \, d\theta = -\frac{1}{2} \cos 2\theta \]

14. \[\int_\gamma \frac{1}{z} \, dz = 0\]

The closed curve \(\gamma\) will equal zero if \(\gamma\) lies within a domain \(A\) on which \(\log z\) has a branch.

- e.g. the standard branch of \(\log z = \ln |z| + \text{arg} z\)

\[-\pi < \text{arg} z < \pi\]

\[\gamma \subset \{ x \in \mathbb{R}, x > 0 \}\]

Since then \[\int_\gamma \frac{1}{z} \, dz = \log z \bigg|_\gamma = 0\]

We will discuss this question precisely in §2.4.
1. a) \[ \int_{-1}^{1} \frac{x^2 + 3}{x^4 + 3x^2 + 1} \, dx = \frac{1}{2} + 3 - \left( \frac{1}{2} - 3 \right) = 0 \]  

b) \( \int_{\gamma} 2z + 3 \, dz = 0 \) because \( \gamma \) is closed and \( 2z + 3 \) has an antiderivative.

d) \( \int_{\gamma} \cos \left( 3 + \frac{1}{z-3} \right) \, dz = 0 \) because \( \gamma \) is contained in \( A = \{ z \in \mathbb{C} : \text{ Re } x < -3 \} \) 
which is simply connected. 

Thus, analytic \( f(z) = \cos \left( 3 + \frac{1}{z-3} \right) \) 
has an antiderivative in \( A \), 
so the contour integral is zero, since \( \gamma \) is closed.

2. \( \int_{\gamma} \frac{1}{z} \, dz = 0 \) for any \( \gamma \) with image in \( A = \{ z \in \mathbb{C} : \text{ Re } z < 0 \} \) 
because \( \frac{1}{z} \) has an antiderivative \( -\frac{1}{z} \).

3. \( \int_{0}^{2\pi} f(z_0 + re^{i\theta}) e^{kr} \, d\theta \)  
if we can write this as a contour integral of some entire 
function around a closed contour, then the value is 0.

Parameterize the circle \( |z - z_0| = r \) (assume \( r > 0 \), although you could also consider \( r = 0, r < 0 \)).

\[ dz = r e^{i\theta} \, d\theta \] 
\[ z - z_0 = r e^{i\theta} \]. 
Thus, \( \int_{0}^{2\pi} f(z_0 + re^{i\theta}) \frac{kr}{e^{i\theta}} \, d\theta = \int_{0}^{2\pi} f(z) \frac{(z - z_0)^{k-1}}{e^{(k-1)i\theta}} \, d\theta \)  
= \frac{1}{e^{i\theta}} \int_{\gamma} f(z) \frac{(z - z_0)^{k-1}}{e^{i\theta}} \, dz = k = 1, 2, \ldots 
this sum is entire, so the contour integral is 0.

4. A "looks" simply connected, 
so we know a branch \( g \) of \( \log z \) exists.

More explicitly:

6. You could parameterize, but 

\[ \int_{\gamma} z - \frac{1}{z} \, dz = \frac{2\pi}{i} - \log z \] 
\[ = -\frac{1}{i} - (\text{ln}(1 + i\pi/2) - i(\text{ln}(1 + i\pi/2) + i(\pi/2)) \] 
\[ = -1 - i \frac{\pi}{2} \]
Solution to class exercise:

\[ Y(t) = y(h(t)) \]
\[ X(t) = \tilde{y}(h(t)) \]

\[ \int_{a}^{b} f(z) \, dz = \int_{c}^{d} f(\tilde{y}(h(t))) \tilde{y}'(h(t)) \, dt \quad \forall \]

is a real calc I integral + \( i \) times another real calc I integral.

so usual substitution works.

\[ t = h(t) \quad \Rightarrow \quad \frac{dt}{dh} = h'(t) \]
\[ d\tilde{y} = h'(t) \, dt \]

\[ x = \int_{k(c)}^{k(d)} f(\tilde{y}(h(t))) \tilde{y}'(h(t)) \, dt \]

**Case I:** \( k(c) = a \quad \text{and} \quad k(d) = b \)
\[ x = \int_{a}^{b} f(\tilde{y}(h(t))) \tilde{y}'(h(t)) \, dt = \int_{a}^{b} f(z) \, dz \]

**Case II:** \( k(c) = b \quad \text{and} \quad k(d) = a \)
\[ x = \int_{b}^{a} f(\tilde{y}(h(t))) \tilde{y}'(h(t)) \, dt = -\int_{a}^{b} f(x) \, dx \]
\[ = -\int_{a}^{b} f(z) \, dz \]