1.3 1a) \( e^{2\cdot i} = e^2 e^{i\cdot 1} = e^2 (\cos 1 + i \sin 1) = e^2 \cos 1 + i e^2 \sin 1 \)

4b) \( \sin z = 4 \)
\[
\frac{1}{2i}(e^{i\cdot z} - e^{-i\cdot z}) = 4
\]

write \( \omega = e^{i\cdot z} \rightarrow \omega - \frac{1}{\omega} = 8i \)
\[
\omega^2 = 8i \cdot \omega - 1 = 0
\]

\[
\omega = \frac{8i \pm \sqrt{-64 + 4}}{2} = 4i \pm \sqrt{15} \cdot i
\]

\( w = e^{i\cdot z} = e^{(x+iy)} \) (\( z = x+iy \))

\[\frac{\partial}{\partial z} \ln \frac{
\frac{\partial}{\partial z} \ln (4 \pm \sqrt{15}i)
\]}

\( 1 \) \( \Rightarrow \) \( e^{-y} = (4 \pm \sqrt{15}) \)
\[\Rightarrow y = -\ln (4 \pm \sqrt{15}) \]

\( \arg (\theta) \Rightarrow x = \frac{\pi}{2} + 2\pi n, n \in \mathbb{Z} \) (since \( \arg (\frac{\pi}{2}) = \frac{\pi}{2} \))

So
\[
\theta = \left\{(\frac{\pi}{2} + 2\pi n) - \ln (4 \pm \sqrt{15})i \right\}
\]

5a) \( \log 1 = \ln 1 + i \cdot \arg 1 \)
\[= 0 + i \cdot (0 + 2\pi n), n \in \mathbb{Z} \]
\[\log 1 = \left\{(2\pi n) i \right\}, n \in \mathbb{Z} \]

6a) \( \log (-i) = \ln |-1| + i \cdot \arg (-i) \)
\[
\frac{1}{-i + e^{-\pi/2}} = 0 + i (-\frac{\pi}{2} + 2\pi n), n \in \mathbb{Z}
\]
\[\ln (-i) = \left\{(-\frac{\pi}{2} + 2\pi n)i \right\}, n \in \mathbb{Z} \]

7a) \( (-i)^z = e^{z \cdot \log (-i)} \)
\[w := e^{\log z} \]
\[
e^i \left( \ln |-1| + i \cdot \arg (-i) \right) \quad (\text{hey, we just did this in 6a!})
\]
\[
e^i \left( -\frac{\pi}{2} + 2\pi n \right) \cdot i, n \in \mathbb{Z}
\]
\[
\left\{ e^{\frac{\pi}{2} + 2\pi n} \right\}
\]
\[\text{same as } \left\{ e^{\frac{\pi}{2} - 2\pi n} \right\}, n \in \mathbb{Z} \]

9) \( (e^{i\cdot z}) = e^{i\cdot (x+iy)} = e^{-y} (\cos x + i \sin x) = e^{i\cdot y} e^{-ix} = e^{i\cdot z} \)

So, the question is really asking,
when does \( e^{i\cdot z} = e^{i\cdot z} \)
\[\text{iff } -y - ix = e^{i(x+iy)} \quad \text{iff } e^{-y - ix} = e^{i\cdot y + i\cdot x} \quad (\theta) \text{ iff } y = 0 \]
\[x = n\pi \quad n \in \mathbb{Z} \]

\[\theta = \left\{ n\pi \right\}, n \in \mathbb{Z} \]
10. In this problem we choose $0 \leq \theta < 2\pi$ to write $z = re^{i\theta}$ ($r > 0$) and define $\sqrt{z} := \sqrt{r} e^{i(\theta/2)}$ as our "branch" of the square root function.

When does $\sqrt{z^2} = z$?

If $0 \leq \theta < \pi$ and $z = re^{i\theta}$, then $z^2 = r^2 e^{i(2\theta)}$ and $0 \leq 2\theta < 2\pi$ (i.e., for $z = x+iy$ with $y > 0$ or $x < 0$ and $y = 0$). So $\sqrt{z^2} = \sqrt{r^2} e^{i(2\theta/2)} = re^{i\theta} = z$.

If $\pi \leq \theta < 2\pi$, and $z = re^{i\theta}$, then $z^2 = r^2 e^{i(2\theta - 2\pi)}$, and $\pi \leq \theta < 2\pi$ if $2\pi < 2\theta < 4\pi$ or $0 < 2\theta - 2\pi < 2\pi$.

So $\sqrt{z^2} = re^{i(\frac{2\theta - 2\pi}{2})} = re^{i(\theta - \pi)} = -z$.

16b): $\sinh z := \frac{1}{2} (e^{-z} - e^z)$, $\cosh z := \frac{1}{2} (e^{-z} + e^z)$

$\sinh (2z) = \frac{1}{2} (e^{2z} - e^{-2z})$ whereas $\sinh z, \cosh z, \sinh 2z, \cosh 2z, \sinh z^2 = \frac{1}{2} (e^{-z} - 2z) + \frac{1}{2} (e^z - 2z) + \frac{1}{2} (e^{-z} - 2z) + \frac{1}{2} (e^z - 2z)$

$= \frac{1}{4} \left[ e^{2z} + e^{-2z} - 2z - 2z - 2z - 2z \right] = e^{2z} + e^{-2z} + e^{2z} + e^{-2z} - 2z - 2z - 2z - 2z = \frac{1}{2} \left[ e^{2z} + e^{-2z} - 2z - 2z \right] = \sinh 2z$.

21) If $z = e^{i\theta}$, then $2 + \frac{1}{2} = e^{i\theta} + e^{-i\theta} = 2 \cos \theta$ varies between $-2$ and $2$ as $0 \leq \theta \leq \pi$ and $-\pi < \theta < 0$ usually.

Thus the mapping $z + \frac{1}{2}$ maps the unit circle $(2 + 1)$ onto $[-2, 2]$ since each $-2 < x < 2$ can be written as $2 \cos (\theta), 2 \cos (-\theta)$ for some $0 < \theta < \pi$.

23) If $z = re^{i\theta}$, then $\frac{1}{2} = \frac{r}{2} e^{-i\theta}$.

Thus if $|z| = r$, $|\frac{1}{2}| = \frac{1}{2}$, so if $|z| > 1$, $\frac{1}{2} |< 1$, and vice versa.

This also shows that if $\arg (z) = \theta$, $\arg (\frac{1}{2}) = -\theta$.

For $\theta$ fixed, the ray $\{te^{i\theta} \}$ is mapped to the ray $\{se^{-i\theta} \}$ ($s > 0$, $t > 0$) so rays with constant argument are mapped to their reflections across the x-axis.
3b) By addition angle, \[ \cos(\theta + \omega) = \cos \theta \cos \omega - \sin \theta \sin \omega \]
so \[ \cos(\theta + 2\pi) = \cos \theta \cos 2\pi - \sin \theta \sin 2\pi \]
\[ \cos 2\pi = 1, \sin 2\pi = 0, \text{ since \, \omega \text{ is complex} } \]
\[ \text{Cosine \, \text{function extends the real variable \, \cos} \]
\[ \text{So \, \cos(\theta + 2\pi) = \cos \theta \text{ \, \forall \, } \theta \in \mathbb{C} \]
Since \, 2\pi \text{ is the minimum period when restricting to real } x,
the period for \, x \in \mathbb{C} \text{ can be no shorter than } 2\pi \text{.} \quad \blacksquare

\[ \begin{align*}
91.4 
\text{ a) Write } W = u + iv, \quad \text{Re} \, W = u, \text{ Im} \, W = v \\
\text{ then } |W| = \sqrt{u^2 + v^2} = \sqrt{u^2 + v^2} = |u| \quad \blacksquare
\text{ b) } |v| = \sqrt{v^2} = \sqrt{u^2 + v^2} = |u| \quad \blacksquare
\text{ c) } |u| \leq |u| + |v| \text{ iff } |u|^2 + |v|^2 \leq |u|^2 + |v|^2 + 2|uv| \\
\text{ which is true} \quad \blacksquare
\]

2b) This is a special case of the 3220 theorem that a vector valued function
has a limit as \( x \to x_0 \) iff each component function does.

Write \( f(z) = u(x,y) + iv(x,y) \)
and let \( a + bi \) be any complex number (which will eventually represent the limit)
We consider \( f(z) - (a + bi) = (u(x,y) - a) + (v(x,y) - b)i \)

Consider the inequalities (from 91):
\[ |u(x,y) - a| \leq |f(z) - (a + bi)| \leq |u(x,y) - a| + |v(x,y) - b| \]
The (1a) inequalities imply that if \( f(z) \to a + bi \) as \( z \to z_0 \)
then also \( u(x,y) \to a \) and \( v(x,y) \to b \).

The (1c) inequality implies that if \( u(x,y) \to a \) and \( v(x,y) \to b \) as \( z \to z_0 \)
then also \( f(z) \to a + bi \).

If \( f(z) \) is continuous at \( z_0 \) then \( f(z_0) = a + bi = u(x_0,y_0) + iv(x_0,y_0) \)
so \( a = u(x_0,y_0) \) and \( b = v(x_0,y_0) \), so \( x \) implies \( u \& v \)
are continuous at \( (x_0, y_0) \)

If \( u, v \) are cont at \((x_0, y_0)\), with \( u(x_0, y_0) = a \) \& \( v(x_0, y_0) = b \),
then \( (\star \star) \) implies \( f(z) \) is continuous at \( z_0 \). \quad \blacksquare
3) \( f \) cont, \( f(z_0) \neq 0 \) \( \Rightarrow \) \( \exists \) neighborhood \( \delta \) \( \ni \) \( z_0 \) on which \( f(z) \neq 0 \)

**Proof:**

Let \( \varepsilon = m > 0 \).

For \( \varepsilon = m \) \( \exists \delta > 0 \) s.t.

\[
|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon = m
\]

For such \( \delta \),

\[
|f(z) - f(z_0)| > |1| |f(z_0)| - |f(z) - f(z_0)| \quad \text{reverse triangle inequality}
\]

4) \( \{z_0\} \) closed iff \( \mathbb{C} \setminus \{z_0\} \) open.

Let \( z \in \mathbb{C} \setminus \{z_0\} \).

Then \( |z - z_0| = m > 0 \).

So if \( |w - z| < m \) then \( |w - z_0| = |(w - z) + (z - z_0)| 

\[
> |z - z_0| - |w - z| \quad \Rightarrow \quad m - m = 0
\]

5) A finite union of points is closed because a single point is \((4)\), and because closed sets are closed under finite unions. Thus the complement of a finite number of points is open.

8) Show \( f(z) = 1/z \) is cont at \( z_0 \).

Let \( \varepsilon > 0 \) be given. Pick \( \delta = \varepsilon \).

By reverse \( \Delta \),

\[
|z - z_0| > |z_0 - z_0| \quad \Rightarrow \quad |z_0 - z_0| > 1/z_0 - 1/z_0
\]

Thus, \( |z - z_0| < \delta \)

\[
\Rightarrow \quad |z_0 - z_0| < \varepsilon
\]

11) If \( |z_1| < 1 \) then \( \{n z_1^n\} \to 0 \)

**pf:**

This is equivalent to proving \( |n z_1^n| = n |z_1|^n \to 0 \).

In fact, for \( 0 \leq r < 1 \) the series \( \sum_{n=1}^{\infty} n |z_1|^n \) converges by the ratio test:

\[
\frac{(n+1) |z_1|}{} = \frac{n+1}{} r < 1.
\]

Thus the individual terms approach 0.

If \( |z_1| < 1 \), \( n |z_1|^n \to 0 \) so no sequence doesn't converge to a finite limit.

If \( |z_1| > 1 \), \( n |z_1|^n \to 0 \) as well, so also doesn't converge to a finite limit.

13 a) \( \{z \ s.t. \ |z| < 1\} = D(z, 1) \) is open. It is not closed (because e.g. its complement is not open.)

b) \( \{z \ s.t. \ 0 \leq |z| < 1\} \) not open. Also not closed.

c) \( \{z \ s.t. \ |z| \leq 1\} \) is closed but not open.
16 a) \( \{ z \mid 1 < \text{Re} z < 2 \} \)
   is not compact (neither closed nor bounded)
   and is connected (because pathwise connected).

b) \( \{ z \mid 2 \leq |z| \leq 3 \} \) is compact (closed & bd)
   and connected (pathwise connected).

c) \( \{ z \mid \Re z \leq 5 \text{ and } |\Im z| > 1 \} \)
   is compact (closed & bd)
   but not connected
   (If the set is called \( A \),
    then \( A \cap \{ z \mid |\Im z| < \frac{1}{2} \} \) is a disconnection.)

18. \( f: \mathbb{C} \rightarrow \mathbb{C} \) is cont if \( \{ z_n \} \rightarrow z_0 \) in \( \mathbb{C} \) implies \( f(z_n) \rightarrow f(z_0) \).

\( \Rightarrow \text{pf:} \) Let \( f: \mathbb{C} \rightarrow \mathbb{C} \) be continuous.
Let \( \{ z_n \} \rightarrow z_0 \).
Let \( \varepsilon > 0 \)

then cont. at \( z_0 \) so \( \exists \delta \) s.t. \( |z-z_0| < \delta \Rightarrow |f(z)-f(z_0)| < \varepsilon \).

\( \Rightarrow \text{pf:} \) we show the logically equivalent statement that \( f \) not cont on \( A \) \( \Rightarrow \exists z_0 \in A \), \( \{ z_n \} \rightarrow z_0 \) and \( f(z_n) \neq f(z_0) \).

\( \Rightarrow \text{pf:} \) if \( f \) is not cont on \( A \)

\( \exists z_0 \text{ s.t. } z_0 \in A \), \( f \) not cont at \( z_0 \).

i.e. \( \exists \varepsilon > 0 \) s.t. \( \forall \delta \text{ s.t. } |z-z_0| < \delta \Rightarrow |f(z)-f(z_0)| \geq \varepsilon \).

i.e. \( \exists \varepsilon > 0 \text{ s.t. } \forall z \in A \) \( \exists z \) with \( |z-z_0| < \delta \) but

\( |f(z)-f(z_0)| \geq \varepsilon \).

So, for \( \delta = \frac{1}{n} \) (n \( \in \) \( \mathbb{N} \)),
pick \( z_n \in A \) s.t. \( |z_n-z_0| < \frac{1}{n} \)
and \( |f(z_n)-f(z_0)| \geq \varepsilon \).

Thus \( \{ z_n \} \rightarrow z_0 \) (by construction),
and \( \{ f(z_n) \} \not\rightarrow f(z_0) \)

(since for the given \( \varepsilon \)
\( \forall n \text{ s.t. } n \in \mathbb{N} \)
\( \Rightarrow |f(z_n)-f(z_0)| \geq \varepsilon \).