Math 2270-1
Sample Final Exam
SOLUTIONS
December 8, 2000

This exam is closed-book and closed-note. You may not use a calculator which is capable of doing linear algebra computations. In order to receive full or partial credit on any problem, you must show all of your work and justify your conclusions. There are 200 points possible, and the point values for each problem are indicated in the right-hand margin. Of course, this exam counts for 30% of your final grade even though it is scaled to 200 points. Good Luck!

1) Let

\[
A := \begin{bmatrix}
1 & -1 & 2 \\
1 & 0 & 1
\end{bmatrix}
\]

Then \(L(x) = Ax\) is a matrix map from \(\mathbb{R}^3\) to \(\mathbb{R}^2\).

1a) Find the four fundamental subspaces associated to the matrix \(A\).

(20 points)

```maple
restart; with(linalg):
Warning, the protected names norm and trace have been redefined and unprotected

> A := matrix(2, 3, [1, -1, 2, 1, 0, 1]);

> rref(A);

So, the row space is the span of \([[1,0,1],0,1,-1]]\). The column space is the span of the first two columns of \(A\), and this is all of \(\mathbb{R}^2\). The nullspace can be obtained by backsolving the homogeneous system, using rref(A). We see that \(x_3=t\), \(x_2=t\), \(x_1=-t\), so \(x=t[-1,1,1]\), so a nullspace basis is \([-1,1,1]\). Since the column space is all or \(\mathbb{R}^2\), its orthogonal complement is the zero vector.

1b) State and verify the theorem which relates rank and nullity of \(A\), in this particular case

\(rank + nullity = \#cols(A)\). In this case \(2+1=3\).
1c) Find an orthonormal basis for $\mathbb{R}^3$ in which the first two vectors are a basis for the rowspace of $A$ and the last vector spans its nullspace.

The rowspace and nullspace of $A$ are already orthogonal, so I just need to Gramschmidt my rowspace basis, and then normalize:

```
> w1 := vector([1, 0, 1]);
w1 := [1, 0, 1]
> w2 := vector([0, 1, -1]);
w2 := [0, 1, -1]
> v1 := evalm(w1);
v1 := [1, 0, 1]
> v2 := evalm(w2-(dotprod(w2, v1)/dotprod(v1, v1))*v1);
v2 := [1/2, 1/2]
```

So my basis is

```
[ [1/2*sqrt(2), 1/6*sqrt(6), -1/3*sqrt(3)]
 [0, 1/3*sqrt(6), 1/3*sqrt(3)]
 [1/2*sqrt(2), -1/6*sqrt(6), 1/3*sqrt(3)]
]
```

1d) Find an implicit equation $ax+by+cz = 0$ satisfied for precisely those points which are in the rowspace of $A$.

I can take normal vector as my nullspace basis, so the equation is $-x+y+z=0$.

2a) Exhibit the rotation matrix which rotates vectors in $\mathbb{R}^2$ by an angle of $\alpha$ radians in the counter-clockwise direction.

```
rot := alpha -> [cos(alpha), -sin(alpha), sin(alpha), cos(alpha)]
```

2b) Verify that the product of an $\alpha$-rotation matrix with a $\beta$-rotation matrix is an $(\alpha+\beta)$-rotation matrix.

```
[cos(alpha) -sin(alpha), cos(beta) -sin(beta)]
[+]
[sin(alpha) cos(alpha), sin(beta) cos(beta)]
```

```
By trig addition angle formulas we recognize this last matrix as

\[
\begin{bmatrix}
\cos(\alpha + \beta) & -\sin(\alpha + \beta) \\
\sin(\alpha + \beta) & \cos(\alpha + \beta)
\end{bmatrix}
\]

3) Find the least-squares line fit through the four points \((1,1), (2,1), (-1,0), (-2,0)\) in the plane. (10 points)

\[
A := \begin{bmatrix}
1 & 1 \\ 2 & 1 \\ -1 & 1 \\ -2 & 1
\end{bmatrix}
\]

\[
b := \begin{bmatrix}
1 \\ 1 \\ 0 \\ 0
\end{bmatrix}
\]

Our least squares solution will solve

\[
\begin{bmatrix}
1 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 \\ -2 & 1
\end{bmatrix}
\begin{bmatrix}
m \\ b
\end{bmatrix} =
\begin{bmatrix}
1 \\ 1 & 2 & -1 & -2 \\ 1 & 1 & 1 & 1 \\ 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
10 & 0 & m \\ 0 & 4 & b
\end{bmatrix} =
\begin{bmatrix}
3 \\ 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
m \\ b
\end{bmatrix} =
\begin{bmatrix}
\frac{3}{10} \\ \frac{1}{2}
\end{bmatrix}
\]

So the least squares line fit is

\[
y = \frac{3}{10} x + \frac{1}{2}
\]
4a) Explain the procedure which allows one to convert a general quadratic equation in n-variables

\[ x^T A x + B x + c = 0 \]

into one without any “cross terms”. Be precise in explaining the change of variables, and the justification for why such a change of variables exists.

(10 points)

A is symmetric, so we can diagonalize it with an orthogonal matrix P, i.e. \( \text{transpose}(P)AP = \text{Lambda} \). We then let

\[ x = Py \]

and the quadratic form transforms to

\[ y^T \text{Lambda} y + B P y + c = 0 \]

4b) Apply the procedure from part (4a) to put the conic section

\[ 6 x^2 + 9 y^2 - 4 xy + 4 \sqrt{5} x - 18 \sqrt{5} y = 5 \]

into standard form. Along the way, identify the conic section.

(20 points)

\[ A := \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix} \]

\[ > \text{eigenvects}(A); \]

\[ [10, 1, \{[1, -2]\}], [5, 1, \{[2, 1]\}] \]

Since both eigenvalues are positive, we have an ellipse. Continuing, we can take

\[ P := \frac{1}{5 \sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \]

And, using u,v for our new variables, our transformed equation is

\[ 10 u^2 + 5 v^2 + 40 u - 10 v = 5 \]

we may complete the square to get

\[ 5 (v - 1)^2 - 45 + 10 (u + 2)^2 = 5 \]

\[ 5 (v - 1)^2 + 10 (u + 2)^2 = 50 \]

\[ \frac{1}{10} (v - 1)^2 + \frac{1}{5} (u + 2)^2 = 1 \]

5) Let

\[ C := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \]

5a) Find the inverse of C using elementary row operations.

(15 points)
So the inverse matrix is

\[
\begin{bmatrix}
-1 & -1 & 1 \\
-2 & -1 & 1 \\
4 & 3 & -2
\end{bmatrix}
\]

5b) Find the inverse of C using the adjoint formula.

We calculate that \( \det(C) = 2 - 2 - 1 = -1 \), so the inverse is given by

\[
\frac{1}{\det(C)} \cdot \text{adj}(C)
\]

6a) Define what it means for a function \( L: V \to W \) between vector spaces to be a **linear transformation**.

I) \( L(u+v) = L(u) + L(v) \) for all \( u, v \) in \( V \)

II) \( L(cu) = cL(u) \) for all \( u \) in \( V \), \( c \) in \( \mathbb{R} \).

6b) Define what it means for a set \( S = \{v_1, v_2, \ldots, v_n\} \) to be a **basis** for a vector space \( V \).
S is a basis for V if it is linearly independent and spans V.

6c) For a linear map L as in part (6a), define the **nullspace (kernel)** of L.

**The kernel of L is the set of all vectors in V which satisfy L(v)=0.**

6d) Let S={v_1, v_2, ..., v_n} be a basis for V, and T={w_1, w_2, ..., w_m} be a basis for W. Let L:V→W be linear. Explain what the matrix for L with respect to S and T is, and how to compute it.

**The matrix for L with respect to S and T is the matrix A which has the property that A times the S-coordinates of a vector v in V equals the T-coordinates of L(v), for all v in V. The jth column of A is just the coordinates of L(v_j), with respect to T. (So that’s one way to compute it.)**

7) Let

\[
A := \begin{bmatrix}
-7 & -6 & 5 \\
4 & 4 & -2 \\
-6 & -5 & 6
\end{bmatrix}
\]

7a) Let

\[
T := \begin{bmatrix}
1 & -1 & 0 \\
0 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}
\]

(Note these are the columns of your matrix C from #5.) Let L(x)=Ax, a matrix map from R^3 to R^3. Thus A is the matrix of L with respect to the standard basis of R^3. Find the matrix of L with respect to the T basis.

**This means use the T-basis in both the domain and range space.**

**Method I: The matrix for L with respect to T is the triple product \([P_{T→E}]A[P_{E→T}]\)**

\[
> A := \text{matrix}(3, 3, [-7, -6, 5, 4, 4, -2, -6, -5, 6]);
\]

\[
A := \begin{bmatrix}
-7 & -6 & 5 \\
4 & 4 & -2 \\
-6 & -5 & 6
\end{bmatrix}
\]

\[
> \text{PET} := \text{matrix}(3, 3, [1, -1, 0, 0, 2, 1, 2, 1, 1]);
\]

\[
\text{PET} := \begin{bmatrix}
1 & -1 & 0 \\
0 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}
\]

\[
> \text{evalm}(\text{inverse}(<\text{PET}>) * \text{evalm}(A) * \text{evalm}(<\text{PET}>) ;
\]

>
Method 2: For \( T=\{v_1,v_2,v_3\} \) compute \( \{Av_1,Av_2,Av_3\} \) and then find coords with respect to \( T \). It turns out that \( Av_1=3v_1, Av_2=2v_3, Av_3=v_2 \). So it is easy to write down the matrix for \( L \) exhibited above.

In general, this leads to the augmented matrix

\[
\begin{bmatrix}
1 & -1 & 0 & 3 & 0 & -1 \\
0 & 2 & 1 & 0 & 2 & 2 \\
2 & 1 & 1 & 6 & 2 & 1
\end{bmatrix}
\]

From which we take the last 3 columns as our matrix.

7b) Let

\[
v := 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}
\]

Compute \( L(v) \) two ways: once using the matrix for \( L \) with respect to \( T \), and once using the matrix \( A \). Verify that your answers agree.

(10 points)

Method 1: Use the E-coordinates for \( v \)

\[
\begin{bmatrix}
-7 & -6 & 5 \\
4 & 4 & -2 \\
-6 & -5 & 6
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
4
\end{bmatrix}
= \begin{bmatrix}
7 \\
0 \\
13
\end{bmatrix}
\]

Method 2: Use T-coordinates
8a) Let A be an n by n matrix. Then if Ax=Ay it follows that x=y.
\[ \text{FALSE: would only be true if A was non-singular} \]

8b) If A and B are n by n matrices, then
\[ (A + B)^2 = A^2 + 2AB + B^2 \]
\[ \text{FALSE: would only be true if AB=BA} \]

8c) If the vectors \( v_1, v_2, \ldots, v_k \) are orthogonal to \( w \), then any vector in the span of \( \{v_1, v_2, \ldots, v_k\} \) is also orthogonal to \( w \).
\[ \text{TRUE: if the dot product of each vi with w is zero, then the dot product of any linear combination of the vj's with w is also zero.} \]

8d) Every set of five orthonormal vectors in \( \mathbb{R}^5 \) is automatically a basis for \( \mathbb{R}^5 \).
\[ \text{TRUE: orthonormal vectors are automatically linearly independent, and 5 independent vectors automatically span a 5-dimensional space} \]

8e) The number of linearly independent eigenvectors of a matrix is always greater than or equal to the number of distinct eigenvalues.
\[ \text{TRUE: each eigenspace is at least one-dimensional, and the union of eigenspace bases is still linearly independent.} \]

8f) If the rows of a 4 by 6 matrix are linearly dependent then the nullspace is at least three dimensional.
\[ \text{TRUE: rows dependent so row rank is at most 3. Since row rank plus nullity equals 6 this means nullity is at least 3} \]

8g) A diagonalizable n by n matrix must always have n distinct eigenvalues.
\[ \text{FALSE: e.g. the identity matrix has only one distinct eigenvalue (=1), but is diagonal} \]

8h) Every orthogonal matrix is diagonalizable.
\[ \text{FALSE: e.g. rotation matrices have no real eigenvectors (unless the rotation is a multiple of Pi).} \]

8i) If A and B are orthogonal matrices then so is AB.
\[ \text{TRUE: use the fact that the transpose of AB is B transpose time A transpose.} \]

8j) If A is a square matrix and \( A^2 \) is singular, then so is A.
\[ \text{TRUE: if } \det(A^2)=0, \text{ then this also equals } (\det(A))^2 \text{ by multiplicative determinant property, so } \det(A)=0. \]