Math 2250-3  
Monday 10/26

4.5.2 → begin with page 5 Friday.

Example 1: Find all solutions to the 3rd order linear homogeneous differential equation

\[ L(y) = y'''(x) - 3y''(x) - 4y' + 12y = 0 \]

Let's try \( y = e^{rx} \):

\[
\begin{align*}
y' &= re^{rx} \\
y'' &= r^2e^{rx} \\
y''' &= r^3e^{rx}
\end{align*}
\]

\[ L(y) = (r^3 - 3r^2 - 4r + 12)e^{rx} \]

want roots of the characteristic polynomial

possible integer roots divide 12

\[ \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \]

\[ r = 2 : 8 - 12 - 8 + 12 = 0 \checkmark \]

\[
\frac{r^3 - r - 6}{r - 2} = \frac{r^3 - 3r^2 - 4r + 12}{r^3 - 2r^2 - r^2 + 2r - 6r + 12 - 6r + 12 - 6r + 12 - 6r + 12 - 6r + 12}
\]

\[ p(r) = (r - 2)(r^2 - r - 6) = (r - 2)(r + 2)(r - 3) \]

\[ L(y_1) + L(y_2) + L(y_3) = c_1y_1 + c_2y_2 + c_3y_3 \]

\[ = 0 \]

So \( c_1y_1 + c_2y_2 + c_3y_3 \) also satisfies this linear homogeneous DE.

In physics you call this the principle of superposition.

If \( L(y) = f \) then \( L(y + z) = f + g \).

So if \( V \) is our solution space to \( L(y) = 0 \), we see that any

\[ c_1y_1 + c_2y_2 + c_3y_3 \in V. \]

Is \( \text{span}\{y_1, y_2, y_3\} \) all of \( V \)?

Are \( y_1, y_2, y_3 \) linearly independent?

(continue on next page)
Example 2: Solve the initial value problem:

$$\begin{align*}
\begin{cases}
y'''(x) - 3y''(x) - 4y' + 12y = 0 \\
y(0) = 0 \\
y'(0) = -3 \\
y''(0) = 5
\end{cases}
\end{align*}$$

$$\begin{align*}
y(x) &= c_1 y_1 + c_2 y_2 + c_3 y_3 \\
y'(x) &= c_1 y_1' + c_2 y_2' + c_3 y_3' \\
y''(x) &= c_1 y_1'' + c_2 y_2'' + c_3 y_3''
\end{align*}$$

The Wronskian matrix is formed by the solutions and their derivatives. If you have n solutions to a linear homogeneous DE, the matrix for which column j is

$$\begin{pmatrix}
y_j \\
y_j' \\
y_j'' \\
\vdots \\
y_j^{(n-1)}
\end{pmatrix}$$

is called the Wronskian matrix. Its determinant is called the Wronskian, $W(y_1, y_2, \ldots, y_n)$. Could we have solved any IVP for this DE at $x=0$? Could the Wronskian explain why?

\[ A := \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & -2 & 3 & -3 \\ 4 & 4 & 9 & 5 \end{bmatrix} \]

\text{Since } W(y_1, y_2, y_3) \neq 0, \text{ every IVP (at } x=0) \text{ solution is a unique linear combination of } y_1, y_2, y_3. \]
an $n$th order differential operator
\[ \mathcal{L}(y) = P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y \]

is called \textit{linear}, because
\[ \begin{align*}
\mathcal{L}(y_1 + y_2) &= \mathcal{L}(y_1) + \mathcal{L}(y_2) \\
\mathcal{L}(cy) &= c\mathcal{L}(y)
\end{align*} \]

The differential equation
\[ \mathcal{L}(y) = F(x) \]

is \textit{homogeneous} if $F(x) = 0$. Otherwise it is \textit{inhomogeneous}.

\textbf{Theorem 1}: The solution space to the linear, homogeneous, DE
\[ \mathcal{L}(y) = 0 \]
is a vector space, i.e. it is closed under addition and scalar multiplication.
Thus if $y_1, y_2$ are solutions, so is
\[ c_1y_1 + c_2y_2 + \ldots + c_ny_n \]
by * * *.

why: If $y_1, y_2$ are solutions, then $\mathcal{L}(y_1 + y_2) = \mathcal{L}(y_1) + \mathcal{L}(y_2) = 0 + 0$
so $y_1 + y_2$ is.

If $y_1$ is a solution, then
\[ \mathcal{L}(cy_1) = c\mathcal{L}(y_1) = 0, \text{ so } cy_1 \text{ is too.} \]

More generally,
\[ \mathcal{L}(c_1y_1 + c_2y_2 + \ldots + c_ny_n) = c_1\mathcal{L}(y_1) + c_2\mathcal{L}(y_2) + \ldots + c_n\mathcal{L}(y_n) \]
\[ = 0 + 0 + \ldots + 0 = 0. \]

\textbf{Theorem 2}: Let
\[ \begin{align*}
\mathcal{L}(y) &= y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_0(x)y \equiv 0 \\
\text{IVP} & \begin{cases}
y(a) = b_0 \\
y'(a) = b_1 \\
y^{(n-1)}(a) = b_{n-1}
\end{cases} \\
\text{If } P_{n-1}, \ldots, P_0 \text{ are continuous on an interval } I \\
\text{and if } a \in I. \\
\text{Then there is always exactly 1 solution} \\
to this initial value problem. \\
\text{why: makes intuitive sense.} \\
more explanation later in course.
Theorem 3: Let \( \mathbf{X}(y) = y^{(n)} + p_{n-1}y^{(n-1)} + \ldots + p_1 y' + p_0 y \) be as in Theorem 2. Then the solution space to \( \mathbf{X}(y) = 0 \) has dimension \( n \). Thus if we can find \( n \) linearly independent solutions, they will span the solution space.

Why: Using the existence-uniqueness result, Theorem 2, we can find (well, actually, they exist) \( y_1, \ldots, y_n \) solving \( \mathbf{X}(y_j) = 0 \), with initial values

\[
\begin{bmatrix}
y_1(a) \\
y_1'(a) \\
\vdots \\
y_1^{(n-1)}(a)
\end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \begin{bmatrix}
y_2(a) \\
y_2'(a) \\
\vdots \\
y_2^{(n-1)}(a)
\end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots \quad \begin{bmatrix}
y_n(a) \\
y_n'(a) \\
\vdots \\
y_n^{(n-1)}(a)
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
\]

\( \{y_1, \ldots, y_n\} \) are a basis.

(a) span: Let \( y(x) \) solve \( \mathbf{X}(y) = 0 \). Then

\[
y(a) = b_0 \\
y'(a) = b_1 \\
\vdots \\
y^{(n-1)}(a) = b_{n-1}
\]

some \( b_0, \ldots, b_{n-1} \)

compare \( y(x) \) to \( b_0 y_1(x) + b_1 y_2(x) + \ldots + b_{n-1} y_{n-1}(x) \)

these 2 solutions solve the same IVP, so they must be the same function. By Theorem 2, i.e. \( y(x) = b_0 y_1 + b_1 y_2 + \ldots + b_{n-1} y_{n-1} \)

(b) linearly ind: if then

\[
c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0
\]

\[
c_1 y_1' + c_2 y_2' + \ldots + c_n y_n' = 0
\]

\[
\vdots
\]

\[
c_1 y_1^{(n-1)} + \ldots + c_n y_n^{(n-1)} = 0
\]

at \( x = a \) deduce \( c_1 = 0 \), \( c_2 = 0 \), \( \ldots \), \( c_n = 0 \)

Theorem 4: Let \( \mathbf{X}(y) \) be as in Theorems 2 & 3.

Let \( \{y_1, \ldots, y_n\} \) be \( n \) solutions to \( \mathbf{X}(y) = 0 \) on the interval \( I \).

Let \( W = \det \begin{bmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{bmatrix} \) (the Wronskian).

Let \( a \in I \). Then if \( W(a) \neq 0 \), \( y_1, \ldots, y_n \) are a basis.

Why: \( W(a) \neq 0 \) means every IVP at \( a \) has a unique solution expressed as a linear combination of \( y_1, y_2, \ldots, y_n \).
Example 3. Find the solution space to

\[ y'' - 6y' + 9 = 0 \]

**Try** \( y = e^{rx} \)

\[
\begin{align*}
y &= e^{rx} \\
y' &= re^{rx} \\
y'' &= r^2e^{rx}
\end{align*}
\]

\[ L(y) = (r^2 - 6r + 9)e^{rx} = 0 \]

\[ (r - 3)^2 = 0 \]

\[ y_1 = e^{3x} \]

Where's \( y_2 \)?

**Trick:** try \( y_2 = xe^{3x} \)

\[
\begin{align*}
y_2' &= (1 + 3x)e^{3x} \\
y_2'' &= (3 + 3(1 + 3x))e^{3x}
\end{align*}
\]

\[ L(y_2) = e^{3x} \left[ x(9 - 18 + 9) \right] = 0 \]

\[
\begin{bmatrix}
y_1 & y_2 \\
y_1' & y_2'
\end{bmatrix}
= \begin{bmatrix}
e^{3x} & xe^{3x} \\
3e^{3x} & e^{3x}(13x)
\end{bmatrix}
\]

\[ W(y) = \det \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = 1 \]

so \( \{y_1, y_2\} \) is a basis (see Theorem 4, p. 9)

\[ y(x) = c_1e^{3x} + c_2xe^{3x} \]

(note: \( W(x) = e^{3x} \) is never 0).

Example 3 is typical of double roots. If \( p(r) = (r - \alpha)^2 = r^2 - 2\alpha r + \alpha^2 \),

for the 2nd order, linear, homogeneous, constant-coeff DE

\[ y'' - 2\alpha y' + \alpha^2 y = 0 \]

then \( y_1 = e^{\alpha x} \)

\[ y_2 = xe^{\alpha x} \]

are a basis for the solution space.

\[
\begin{align*}
y_2 &= xe^{\alpha x} \\
y_2' &= e^{\alpha x}(\alpha x + 1) \\
y_2'' &= e^{\alpha x}(\alpha + \alpha(\alpha x + 1))
\end{align*}
\]

\[ L(y_2) = e^{\alpha x} \left[ x(\alpha^2 - 2\alpha^2 + \alpha^2) \right] = 0 \]

\[ \left[ +1(-2\alpha + 2\alpha) \right] \]