Def. A second order linear differential equation has the form
\[ A(x) y'' + B(x)y' + C(x)y = F(x) \]
We search for solutions \( y(x) \) on some interval \( [a,b] \).
In this chapter we assume \( A(x) \neq 0 \) on the interval, so the DE could also be written
\[ y'' + p(x)y' + q(x)y = f(x) \]
On reason this differential eqn is called linear is that the "operator" \( \mathcal{L} \)
defined by
\[ \mathcal{L}(y) = y'' + p(x)y' + q(x)y \]
satisfies the linear operator axioms (like matrix mult axioms)
\begin{enumerate}
  \item \( \mathcal{L}(y_1 + y_2) = \mathcal{L}(y_1) + \mathcal{L}(y_2) \)
  \item \( \mathcal{L}(cy) = c \mathcal{L}(y) \)
\end{enumerate}
[we checked these in detail for a particular example on Monday].
Therefore, by vector space theory, the solution set to the homogeneous DE
\[ \mathcal{L}(y) = y'' + p(x)y' + q(x)y = 0 \]
is a subspace. And the general solution to \( \mathcal{L}(y) = f \) is \( y = y_p + y_h \), where
\( y_p \) is any particular solution, and \( y_h \) is the general soltn to the homogeneous equation.

Existence-Uniqueness theorem
If \( p(x), q(x) \) are continuous on the interval \( I \), and \( a \in I \). Then there
is a unique soltn to
\[
\begin{align*}
  \text{IVP:} & \quad \begin{cases} y''(x) + p(x)y' + q(x)y = f(x) \\ y(a) = b_0 \\ y'(a) = b_1 \end{cases} \\
\end{align*}
\]
one reason this makes sense:
\[ y'' = f' - py' - qy \]
so if I know \( y(a), y'(a) \), then I know \( y''(a) \).
But
\[ y'' = f' - py' - py' - qy - qy' \]
so I can find \( y''(a) \).
so I can find all derivs \( y \) at \( a \), i.e. the Taylor series for.
[Of course this would require all coeffs to be infinitely differentiable.]
Consequence of the existence uniqueness theorem:

**Theorem** The solution space to the homogeneous DE

\[ y'' + py' + qy = 0 \]

is 2-dimensional. (So our goal is to find a basis consisting of 2 linearly ind. solutions)

**Proof**: Let \( y_1(x) \) solve \( y'' + py' + qy = 0 \) \{ \begin{align*}
  y_1''(x) &= 0 \\
  y_1'(0) &= 0 \\
  y_1(0) &= 1
\end{align*} \}

\( y_2(x) \) solve \( y'' + py' + qy = 0 \) \{ \begin{align*}
  y_2''(x) &= 0 \\
  y_2'(0) &= 0 \\
  y_2(0) &= 1
\end{align*} \}.

These solutions exist by existence uniqueness theorem, and are unique.

Then the unique solution to

\[
\begin{align*}
  y'' + py' + qy &= 0 \\
  y(a) &= b_0 \\
  y'(a) &= b_1
\end{align*}
\]

is

\[ y(x) = b_0 y_1(x) + b_1 y_2(x) \]

is a soln \( \checkmark \)

\[ y(0) = b_0 \cdot 0 + b_1 \cdot 1 = b_0 \checkmark \]

\[ y'(0) = b_0 \cdot 0 + b_1 \cdot 1 = b_1 \checkmark \]

thus \( \{ y_1(x), y_2(x) \} \) are a basis for the solution space.

(they span by above.

they are independent because

\[ c_1 y_1 + c_2 y_2 = 0 \]

\[ \Rightarrow c_1 y_1' + c_2 y_2' = 0 \]

at \( x = a \) we get

\[ c_1 + 0 = 0 \quad \text{so} \quad c_1 = 0; \]

\[ 0 + c_2 = 0 \quad \text{so} \quad c_2 = 0. \]
Our most important example in the entire course

Model

Newton says

\[ m \ddot{x}(t) = \text{net force} \]

\[ m \ddot{x}(t) = F_s + F_d + F(t) \]

\[ \begin{array}{c}
\text{spring force} \\
\text{drag force} \\
\text{any other applied forces}
\end{array} \]

Good models for spring and drag forces:

If \( F_s = F_s(x) \) (only depends on displacement)

then \( F_s(x) = F_s(0) + \frac{1}{2} F_s''(0) x^2 + \ldots \) (Taylor series).

- For small \( x \), ignore quadratic & above terms
- \( F_s(0) = 0 \) since \( x = 0 \) is equilibrium.

So \( F_s \approx F_s(0) x = -kx \) (coeff should be \( < 0 \) since \( x > 0 \) implies \( F_s < 0 \))

Hoek's "Law"

If \( F_d = F_d(v) \), where \( v = x'(t) \)

then \( F_d(v) \approx F_d(0) + F_d'(0) v + \ldots \)

\[ F_d(v) \approx -cv \] (linear drag)

So \( m \ddot{x}(t) = -kx - cv + F(t) \) is a good model

\[
\begin{cases}
  m \ddot{x}(t) + c \dot{x}(t) + kx = F(t) \\
  x(0) = x_0 \\
  x'(0) = v_0
\end{cases}
\]

In applications

- \( t \): variable
- \( x(t) \): function
Example In the mass-spring system suppose \( m = 2, k = 8, c = 0, \ F(t) = 0 \)

\[
2 x''(t) + 8 x'(t) = 0
\]

\[
x'' + 4 x = 0
\]

\[
x_1(t) = \cos 2t
\]
\[
x_2(t) = \sin 2t
\]

are solutions; check!

They are linearly ind., since if

\[
c_1 x_1 + c_2 x_2 = 0
\]

then

\[
c_1 x_1' + c_2 x_2' = 0
\]

\[
\begin{bmatrix}
x_1 & x_2 \\
x_1' & x_2'
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
\cos 2t & \sin 2t \\
-2 \sin 2t & 2 \cos 2t
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= 0
\]

at \( t = 0 \) ::

\[
\begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= 0
\]

\( c_1 = c_2 = 0 \).

This also shows we can solve any IVP:

\[
\begin{cases}
x'' + 4 x = 0 \\
x(0) = 2 \\
x'(0) = -2
\end{cases}
\]

\[
c_1 x_1(0) + c_2 x_2(0) = 1 \\
c_1 x_1'(0) + c_2 x_2'(0) = -2
\]

-1 \quad 0 \quad 1

\[
1 \quad 0 \\
0 \quad 2
\]

1 \quad 0 \quad \frac{1}{-2}

\[
x(t) = \cos 2t - \sin 2t
\]
how to find a basis for the soln space to

\[ ay'' + by' + cy = 0 \]

where \( a \neq 0, b, c \) are constants.

\[ \begin{align*}
  \text{try} & \quad y = e^{rx} \\
  y' &= re^{rx} \\
  y'' &= r^2e^{rx}
\end{align*} \]

\[ \Rightarrow x(y) = (ar^2 + br + c)e^{rx} \]

set this poly in \( r = 0 \).

\[ \text{called characteristic polynomial} \]

**Example**

\[ y'' - 5y' + 6y = 0 \]

if \( y = e^{rx} \)

\[ (r^2 - 5r + 6)e^{rx} = 0 \]

\[ (r - 3)(r - 2) = 0 \]

\[ r = 2, 3 \]

\[ \begin{align*}
  y_1 &= e^{2x} \\
  y_2 &= e^{3x}
\end{align*} \]

\[ \Rightarrow \text{ lin ind?} \quad \begin{cases}
  c_1y_1 + c_2y_2 = 0 \\
  c_1y_1' + c_2y_2' = 0
\end{cases} \]

\[ \begin{bmatrix}
  y_1 & y_2 \\
  y_1' & y_2'
\end{bmatrix} \begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix} = 0 \]

\[ \begin{bmatrix}
  e^{2x} & e^{3x} \\
  2e^{2x} & 3e^{3x}
\end{bmatrix} \]

\[ \text{at } x = 0 \quad \begin{bmatrix}
  1 & 1 \\
  2 & 3
\end{bmatrix} \quad \text{det} = 3 - 2 = 1 \]

\[ \Rightarrow c_1 = c_2 = 0 \]

In general,

\[ W = \det \begin{bmatrix}
  y_1 & y_2 \\
  y_1' & y_2'
\end{bmatrix} \quad \text{is called the Wronskian} \]

if it is \( \neq 0 \) anywhere, solutions are l.i.

in an example

\[ W = \left| \begin{bmatrix}
  e^{2x} & e^{3x} \\
  2e^{2x} & 3e^{3x}
\end{bmatrix} \right| = e^{5x} \neq 0 \quad \text{forall } x. \]