Recall from last Wednesday our examples and definitions.

A **linear combination** of the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \) is any vector of the form

\[
\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_k \vec{v}_k
\]

The coefficients \( c_1, c_2, \ldots, c_k \) are called the **linear combination coefficients**.

The **span** of \( \{\vec{v}_1, \ldots, \vec{v}_k\} \) is the collection of all linear combinations of \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \).

We were also interested in whether linear combination coefficients are unique, and this leads to the following definitions (see below):

The collection \( \{\vec{v}_1, \ldots, \vec{v}_k\} \) of vectors is

- **linearly independent** if the only way
  \[
  c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_k \vec{v}_k = \vec{0}
  \]
  is if \( c_1 = c_2 = \ldots = c_k = 0 \).

- **linearly dependent** if there is some choice of linear combo coefficients which are not all zero, but so that
  \[
  c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_k \vec{v}_k = \vec{0}.
  \]

**nomenclature**:

- another way to characterize linearly dependent is that at least one of the \( \vec{v}_i \)'s is a linear combination of the other \( \vec{v}_i \)'s.

**Theorem** (to tie these notions in with last Wednesdays lecture)

The linear combination coefficients of each \( \vec{w} \in \text{span}\{\vec{v}_1, \ldots, \vec{v}_k\} \)

are unique if and only if \( \{\vec{v}_1, \ldots, \vec{v}_k\} \) is linearly independent.

**proof**: If \( \vec{w} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \ldots + a_k \vec{v}_k \)

\[
\vec{w} = b_1 \vec{v}_1 + b_2 \vec{v}_2 + \ldots + b_k \vec{v}_k
\]

then \( \vec{0} = \vec{w} - \vec{w} = (a_1 - b_1) \vec{v}_1 + (a_2 - b_2) \vec{v}_2 + \ldots + (a_k - b_k) \vec{v}_k \)

\[
= c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots + c_k \vec{v}_k \quad (c_j = a_j - b_j)
\]

So if \( \{\vec{v}_1, \ldots, \vec{v}_k\} \) is linearly independent each \( c_j = 0 \), so each \( a_j = b_j \)

and the linear combo coefficients of \( \vec{w} \) are unique.

Conversely, if linear combo coefficients are unique, then the only way to express \( \vec{0} \) is \( \vec{0} = 0 \vec{v}_1 + 0 \vec{v}_2 + \ldots + 0 \vec{v}_k \).

So \( \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \) are linearly independent.
§ 4.3 #1: Are \( \vec{v}_1 = \begin{bmatrix} 4 \\ -2 \\ 6 \\ -4 \end{bmatrix} \), \( \vec{v}_2 = \begin{bmatrix} 6 \\ -3 \\ 9 \\ -6 \end{bmatrix} \) linearly independent or dependent?

What geometric object is \( \text{span}\{\vec{v}_1, \vec{v}_2\} \)?

§ 4.3 #3: Are \( \vec{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \), \( \vec{v}_2 = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \), \( \vec{v}_3 = \begin{bmatrix} 7 \end{bmatrix} \) linearly independent or dependent?

What geometric object is \( \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \)?

Last Wednesday we quickly discussed the following important questions:

Can more than \( n \) vectors in \( \mathbb{R}^n \) be linearly independent?

Why not?

Can less than \( n \) vectors in \( \mathbb{R}^n \) span all of \( \mathbb{R}^n \)?

Why not?

If you have exactly \( n \) vectors in \( \mathbb{R}^n \) what tests determine whether they are linearly independent?

whether they span \( \mathbb{R}^n \)?
There are objects other than vectors in $\mathbb{R}^n$ which one can add and scalar multiply, and for which the expected arithmetic rules apply. Thus we will be able to consider concepts like "span" and linear independence/dependence in these other settings as well.

The main example to consider here is

$$\mathcal{F} = \text{the set of real-valued functions} = \{ f: \mathbb{R} \to \mathbb{R} \},$$

with domain $\mathbb{R}$

where function addition and scalar multiplication are defined as in Calculus:

$$\begin{align*}
(f + g)(x) &= f(x) + g(x) \\
(c \cdot f)(x) &= c \cdot f(x)
\end{align*}$$

In $\mathbb{R}^n$ and in $\mathcal{F}$, addition and scalar mult satisfy the vector-space axioms:

As a set $V$ of "vectors", together with operations $+$, scalar multiplication is called a vector space if the following axioms hold

(a) whenever $\vec{u}, \vec{v} \in V$ then $\vec{u} + \vec{v} \in V$ (closure w.r.t addition)

(b) whenever $\vec{u} \in V$, $k \in \mathbb{R}$, then $k \cdot \vec{u} \in V$ ("scaler multiplication")

(a) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ \hspace{1cm} \text{commutative}

(b) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ \hspace{1cm} \text{associative}

(c) $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ \hspace{1cm} \text{zero vector exists in } V

(d) $\vec{u} + (-\vec{u}) = \vec{0}$ \hspace{1cm} \text{additive inverses exist}

(e) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ \hspace{1cm} \text{distributive prop}

(f) $(a+b)\vec{u} = a\vec{u} + b\vec{u}$

(g) $a(b\vec{u}) = (ab)\vec{u}$

(h) $1 \cdot \vec{u} = \vec{u}$

• What is the zero "vector" in $\mathcal{F}$?

• Are the "vectors" $\{1, x, x^2\}$ linearly independent in $\mathcal{F}$?

What is their span?
Examples of vector spaces

1. \( V = \mathbb{R}^n \) as we've been doing.
2. \( \mathcal{F} = \text{real valued fns with domain } \mathbb{R} \)
   - we will use this vector space a lot when we return to differential eqns.
3. Subspaces \( W \) of a vector space \( V \)

A subset of \( V \) that is a vector space itself, via the \( V \) operations.

To check whether \( W \) is a subspace, you need only check
(a) closure wrt +
(b) closure wrt scalar mult.

Then (a)-(b) are basically inherited from \( V \).

Important subspaces

- The solution set \( W \) to a homogeneous matrix equation \( A\mathbf{x} = \mathbf{0} \)
  r.e. \( \{ \mathbf{x} \in \mathbb{R}^n \text{ s.t. } A\mathbf{x} = \mathbf{0} \} \)
  for a given \( A \in \mathbb{R}^{m \times n} \).

Check this is a subspace of \( \mathbb{R}^n \):
(a) If \( \mathbf{x}, \mathbf{y} \in W \) then \( A\mathbf{x} = \mathbf{0}, A\mathbf{y} = \mathbf{0} \)
so \( A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} \). Hence \( \mathbf{x} + \mathbf{y} \in W \).
(b) If \( \mathbf{x} \in W \) then \( A\mathbf{x} = \mathbf{0} \)
so \( A(k\mathbf{x}) = kA\mathbf{x} = \mathbf{0} \) too.
so \( k\mathbf{x} \in W \).

- \( W = \text{span}\{v_1, v_2, \ldots, v_m\} \).

(a) If \( \mathbf{x} = c_1 v_1 + \ldots + c_m v_m \) then \( \mathbf{x} + \mathbf{y} = (c_1 + d_1)v_1 + \ldots + (c_m + d_m)v_m \in W \).
(b) If \( \mathbf{x} = c_1 v_1 + \ldots + c_m v_m \) then \( k\mathbf{x} = (c_1 k)v_1 + \ldots + (c_m k)v_m \in W \).

Try some hw from 6.4.2, e.g. 6, 9, 15.