Exam review session
(we'll go over practice exam, mainly.)
EMCB 112 tomorrow
(Sat Oct 2)
10:00-11:30 a.m.

Math 2250
Fri Oct 1.

Finish determinants,
and nifty formulas for $A^{-1}$
and Cramer's rule for solving $Ax = b$.

Recall recursive def of det:

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{(expansion across row } i \text{ of } A)$$

$$= \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{(expansion down col } j \text{ of } A).$$

- expanding down any column, or across any row
  yields same value; i.e. $|A|$.  
- If $A$ is (upper or lower) triangular,
  $|A|$ is the product of its diagonal entries.

**Effects of elementary row operations (or column ops)** on determinants:

1. swapping two rows changes sign of determinant
   - checked this on Wednesday

2. multiplying a row by a constant
   multiplies det by same const:
   $$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ c \cdot a_{i1} & \cdots & c \cdot a_{in} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = c \sum_{j=1}^{n} (c a_{ij}) C_{ij} = c |A|$$

   (the effect of this is that when we do
   row ops, if we factor a "c" out \textt at a row
   we multiply it by the det of what's left
   to get original det.)
(3) replace row $i$ of $A$ with row $i$ of $A + c \cdot $ row $k$ of $A$:

Does not change det $A$!!

reason:

\[
\begin{align*}
\text{row } i & \rightarrow \text{row } i + c \cdot \text{row } k \\
& \rightarrow \begin{vmatrix}
R_1 \\
R_2 \\
\cdots \\
R_i + c \cdot R_k \\
R_n
\end{vmatrix}
\end{align*}
\]

\[
= \sum_{j=1}^{n} (a_{ij} + ca_{kj}) C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} + c \sum_{j=1}^{n} a_{kj} C_{ij}
\]

= $\det A$.

example (did on Wednesday)

\[
\begin{vmatrix}
1 & 2 & -1 \\
0 & 3 & 1 \\
2 & -2 & 1
\end{vmatrix}
\]

= \[
\begin{vmatrix}
1 & 2 & -1 \\
0 & 3 & 1 \\
0 & -6 & 3
\end{vmatrix}
\]

-2$R_1 + R_3$

\[
= \begin{vmatrix}
1 & 2 & -1 \\
0 & 3 & 1 \\
0 & 0 & 5
\end{vmatrix}
\]

+2$R_2 + R_3$

\[
= 3.5 \begin{vmatrix}
1 & 2 & -1 \\
0 & 1 & \frac{1}{3} \\
0 & 0 & 1
\end{vmatrix}
\]

- Row $R_3 / 5$

\[
= 3.5 \begin{vmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{vmatrix}
\]

(Various cleanup row ops, all of "no-change" type)

= 15 \cdot 1 = 15.
**Theorem** \[ \det A \neq 0 \iff \text{ref}(A) = I \iff A^{-1} \text{ exist} \]

- already know this equivalence

**pf:** start with \( A \)

- do elementary row ops
  - change sign if swap rows (original \( \det \) is opposite \( \text{new det.} \))
  - no change if replace row \( i \) by \( row_i + cr \cdot row_k \), \( k \neq i \)
  - if factor \( c \) out of row, original \( \det \) is \( c \) times \( \text{det} \) \( \text{what's left} \)

\[ \text{ref}(A) \]

- if \( \text{ref}(A) = I \)

\[ |A| = c_1c_2...c_N |I| \]

- \( c_i \)'s are \( \pm 1 \), or
- other non-zero constants,
- and \( |A| = c_1c_2...c_N \neq 0 \)

- if \( \text{ref}(A) \neq I \)

\[ |A| = c_1c_2...c_N \begin{bmatrix} \mathbb{I} & \mathbb{W}\end{bmatrix} = 0 \]

**Thm** \[ \det(AB) = \det(A)\det(B) \leftarrow \text{true, but not always:} \]

- \( \det(A + B) \neq \det(A) + \det(B) \) !!!

\[ |A| \]

- \( \text{ref} \)

- \( \text{ref}(A) = I \)

\[ |A| = c_1...c_N |I| \]

\[ c_1...c_N \begin{bmatrix} \mathbb{I} & \mathbb{W}\end{bmatrix} \]

- \( \text{ref}(A) \neq I \)

- \( \text{ref}(A) \) has a row of 0's.

\[ |A| = c_1c_2...c_N \]

- \( \text{ref}(A) \)

- \( \text{ref}(A) \neq I \)

- \( \text{ref}(A) \) has a row of 0's.

- do identical row ops to \( A \)

- \( \text{same as doing row ops to } \)

- \( \text{then multiplying} \)

- \( |AB| = c_1...c_N |I| \)

- \( = |A||B| \)

- \( AB \) = \( c_1...c_N |W|B \)

- \( = |A||B| \)

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- \( = |A||B| \)
If \( B_{mn} = [b_{ij}] \) then the transpose \( B^T \) is defined by

\[
\text{entry}_{ij} (B^T) = \text{entry}_{ji} (B) = b_{ji}
\]

the effect is to switch rows & columns (and vice versa).

e.g. \[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}^T =
\begin{bmatrix}
1 & 4 \\
2 & 5 \\
3 & 6
\end{bmatrix}
\]

**Definition**

the adjoint matrix \( \text{adj}(A) = [C_{ij}] \) (write \( \text{adj}(A) \)) is the transpose of the cofactor matrix \( [C_{ij}] \)

\[
\text{adj}(A) = [C_{ij}]^T
\]

**Theorem**

When \( A^{-1} \) exists,

\[
A^{-1} = \frac{1}{|A|} \text{adj}(A)
\]

**Examples**

\[
A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \quad [M_{ij}] = \begin{bmatrix}
h & e \\
a & d
\end{bmatrix} \quad [C_{ij}] = \begin{bmatrix}
d & -c \\
-b & a
\end{bmatrix} \quad \text{adj}(A) = \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix}
\]

our friend

\[
A = \begin{bmatrix}
1 & 2 & -1 \\
0 & 3 & 1 \\
2 & -2 & 1
\end{bmatrix}
\]

\[
[C_{ij}] = \begin{bmatrix}
1/2 & 1/2 & 1/2 \\
1/2 & -1/2 & 1/2 \\
1/2 & 1/2 & -1/2
\end{bmatrix}
\]

\[
[\text{adj}(A)] = \begin{bmatrix}
5 & 2 & -6 \\
0 & 3 & 6 \\
5 & -1 & 3
\end{bmatrix}
\]

\[
\text{adj}(A) = \begin{bmatrix}
5 & 0 & 5 \\
2 & 3 & -1 \\
-6 & 6 & 3
\end{bmatrix}
\]

\[
A^{-1} = \frac{1}{18} \begin{bmatrix}
5 & 0 & 5 \\
2 & 3 & -1 \\
-6 & 6 & 3
\end{bmatrix}
\] check!
proof that \( A^{-1} = \frac{1}{|A|} \text{adj}(A) \cdot X \): does \( AX = I \)?

\[
\text{entry}_{j,m}(AX) = \sum_{k=1}^{n} a_{j,k} X_{k,m} \cdot \frac{1}{|A|} C_{mh} = \frac{1}{|A|} \sum_{k=1}^{n} a_{j,k} C_{mh} \quad \rightarrow \quad \text{if } l = m \text{ this sum is expansion for } |A| \text{ down } l = m \text{ in col} \]

\[
\text{so entry}_{j,m}(AX) = \frac{|A|}{|A|} = 1 \quad \text{det} = 1
\]

\[
\text{if } l \neq m \text{ we are expanding down the } l \text{th col of a matrix in which we replaced column } m \text{ with the } l \text{th column, so two col's are equal, so det } = 0!
\]

\[
\text{this shows } AX = I.\quad \text{(which implies }XA = I\text{)}.
\]

Cramer's rule

let \( x \) solve \( AX = b \) for invertible \( A \).

then \( x_k = \frac{C_1 C_2 \cdots \hat{b} \cdots C_n}{|A|} \)  

\[
\text{numerator is } \text{matrix obtained from } A \text{ by replacing } k \text{th column } b \text{ by } b. 
\]

proof: \( x_k = \text{entry}_{j,k}(A^{-1}b) \)

\[
= \text{entry}_{j,k} \left( \frac{1}{|A|} \text{adj}(A)b \right) = \frac{1}{|A|} \text{row}_{j,k}(\text{adj}(A)) \cdot b 
\]

\[
= \frac{1}{|A|} \sum_{l=1}^{n} C_{lh} b \quad \text{this is the expansion of the det in numerator of Cramer's rule, down the } k \text{th column}!
\]
Review sheet for 1st exam, which is Monday 10/4.

problem session
Saturday (tomorrow) 10-11:30
EMCB 112

Chapter 1: Methods to solve certain 1st order DE's
integration, for $\frac{dy}{dx} = f(x)$
position, velocity, acceleration, where acceleration is a function of $t$ alone.
separable DE's
growth & decay (exponential), populations, radioactive decay, Newton's law of cooling (with constant ambient temp.)
drug elimination, rumor propagation, disease spread.
NO TERRICELLI ON EXAM
linear 1st order DE's
mixing problems
slope fields and phase diagrams to understand qualitative behavior of solutions
without knowing formula for solution function.
How to draw these, especially for autonomous DE's.

Chapter 2: Applications in depth
population models: logistic, doomsday/extinction, harvesting logistic
understand derivations, how to find solutions (by separating variables),
how to plot slope fields & phase portraits, how to find equilibrium solutions and evaluate stability/instability.
equilibrium solutions & stability for general 1st order autonomous DE's.
acceleration - velocity models, especially linear drag (force proportional to velocity).

NO NUMERICAL METHODS (2A-2.6) ON EXAM.

Chapter 3: Linear systems and matrices
solving linear systems by creating the augmented matrix, using elementary row operations to get row-echelon form, solving solution by backsolving.
geometric meaning of linear systems in 2 or 3 variables
Matrix algebra: addition, scalar multiplication, matrix multiplication
What algebra rules hold, and which one(s) don't.
Matrix inverses
how to compute via row operations
how to solve linear systems with the inverse matrix, if the inverse exists.
Determinants
how to compute with cofactors
how to compute with row operations
Cramer's rule
Adjoint formula for the inverse, esp. $2 \times 2$ & $3 \times 3$ cases.