Laplace transform cont'd.

today:

\[ \mathcal{L}\{t \cos kt\} = \frac{1}{s^2 + k^2} \]

\[ \mathcal{L}\{\cos kt\} = \frac{s}{s^2 + k^2} \]

So \[ \mathcal{L}\{t \cos kt\} = -F'(s) \]

\[ = -\frac{1}{s^2 + k^2} - s(-1)(s^2 + k^2)^{-2}2s \]

\[ = -\frac{s^2 - k^2}{(s^2 + k^2)^2} = \frac{s^2 - k^2}{(s^2 + k^2)^2} \checkmark \]

\[ \mathcal{L}\{t \sin kt\} = \frac{k}{s^2 + k^2} \]

So \[ \mathcal{L}\{t \sin kt\} = -F'(s) \]

\[ = k(-1)(s^2 + k^2)^{-2}2s \]

\[ = \frac{-2ks}{(s^2 + k^2)^2} \checkmark \]

\[ \mathcal{L}^{-1}\left(\frac{1}{(s^2 + k^2)^2}\right) = \mathcal{L}^{-1}\left\{\frac{s^2 + k^2}{(s^2 + k^2)^2} - \frac{(s-k)}{(s^2 + k^2)^2} \right\} \quad \text{(today)} \]

\[ = \frac{1}{2k} \left( \frac{1}{k} \sin kt - t \cos kt \right) \checkmark \]

(useful in resonance problems).
Example  p. 595
\[
\begin{cases}
  x'' + \omega_0^2 x = F_0 \sin \omega t \\
  x(0) = 0 \\
  x'(0) = 0
\end{cases}
\]

resonance (or not) revisited

\[
s^2 X(s) - os - 0 + \omega_0^2 X(s) = \frac{F_0}{s^2 + \omega_0^2}
\]

\[
X(s) \left[ s^2 + \omega_0^2 \right] = \frac{F_0 \omega_0}{s^2 + \omega_0^2}
\]

\[
X(s) = F_0 \omega_0 \left( \frac{1}{s^2 + \omega_0^2} \right) \left( \frac{1}{s^2 + \omega_0^2} \right)
\]

\[
\omega = \omega_0 \quad \text{resonance}
\]

\[
X(s) = \frac{F_0 \omega_0}{(s^2 + \omega_0^2)^2}
\]

\[
x(t) = (\text{from table!})
\]

\[
= \frac{F_0 \omega_0}{2 \omega_0^3} \left( \sin \omega_0 t - \omega_0 t \cos \omega_0 t \right)
\]

\[
x(t) = \frac{F_0}{2 \omega_0^2} \left( \sin \omega_0 t - \omega_0 t \cos \omega_0 t \right) \quad \text{resonance!}
\]

much easier than method of undetermined coefficients!
We could do the spring system problem (Findley notes),

\[
\begin{align*}
\begin{cases}
y''''(t) + 2y'' + y(t) = 4t e^t \\
y(0) = 0 \\
y'(0) = 0 \\
y''(0) = 0 \\
y'''(0) = 0
\end{cases}
\end{align*}
\]

(Example 6 p. 596)

Should get

\[
Y(s) = \frac{4}{(s-1)^2 (s^2+1)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs + D}{s^2 + 1} + \frac{Es + F}{(s^2 + 1)^2}
\]

so

\[
y(t) = Ae^t + Bte^t + C \cos t + D \sin t + E \left(\frac{1}{2} t \sin t\right) + F \left(\frac{1}{2} (\sin t - t \cos t)\right)
\]

so

\[
t \rightarrow \frac{1}{s^2},
\]

\[
te^t \rightarrow \frac{1}{(s-1)^2}
\]

\[
4 = A(s-1)(s^2+1)^2 + B(s^2+1)^2 + (Cs + D)(s^2+1)(s-1)^2 + (Es + F)(s-1)^2
\]

\[
4 = A(s-1)(s^4 + 2s^2 + 1) + B(s^4 + 2s^2 + 1) + (Cs + D)(s^2+1)(s^2-2s+1) + (Es + F)(s^2-2s+1)
\]

Yipes!

> with(inttrans);
> [addtable, fourier, fouriercos, fouriersin, hankel, hilbert, invfourier, invhilbert, invlaplace,
> invmellin, laplace, mellin, savetable];
> ?parfrac;
> ?invlaplace;
> F := s -> 4/( (s-1)^2 * (s^2+1)^2 );
> F := s -> \frac{4}{(s-1)^2 (s^2+1)^2};
> convert(F(s),parfrac,s);
> \frac{2s}{(s^2+1)^2} - \frac{2}{s-1} + \frac{1+2s}{s+1} + \frac{1}{(s-1)^3};
> convert(F(s),parfrac,s);
> invlaplace(F(s),s,t);
> (t+1) \sin(t) + 2 \cos(t) + (-2 + i) e^t.