A differential equation
\[ \frac{dy}{dx} = h(x, y) \]

is called \textit{separable} iff \( h(x,y) \) is a product of a function of \( x \) times a function of \( y \),
\[ \frac{dy}{dx} = g(x) \phi(y) \]

This is equivalent to the DE
\[ \frac{dy}{dx} = \frac{g(x)}{f(y)} \]

where \( f \) and \( \phi \) are reciprocal functions.

**How to solve:**
The algorithm is very simple, but magic: treat \( \frac{dy}{dx} \) as a quotient of differentials (?!), and multiply through to rewrite the DE as
\[ f(y) \, dy = g(x) \, dx \]

Then antidifferentiate the left side with respect to \( y \) and the right side with respect to \( x \).
\[ \int f(y) \, dy = \int g(x) \, dx + C \]

If \( F(y) \) and \( G(x) \) are antiderivatives of \( f(y) \) and \( g(x) \), respectively, then this is the solution
\[ F(y) = G(x) + C \]

This equation defines \( y \) implicitly as a function of \( x \). Sometimes you can use algebra to explicitly solve for \( y \). The constant \( C \) can be adjusted to solve initial value problems.

**Why the method works:**
The use of differentials is disguising an application of the chain rule. Here is the explanation for the magic method: The differential equation
\[ \frac{dy}{dx} = \frac{g(x)}{f(y)} \]

CAN be rewritten as
\[ \text{deqtn2} := f(y) \left[ \frac{dy}{dx} \right] = g(x) \]

If \( y(x) \) is any solution to \( \text{deqtn2} \), then the left side, namely
\[ f(y(x)) \left[ \frac{dy}{dx} \right] \]

is the derivative with respect to \( x \) of
\[ F(y(x)) \]

whenever \( F(y) \) is an antiderivative of \( f(y) \) (with respect to \( y \)). This is just the chain rule! Thus if \( G(x) \) is any antiderivative of \( g(x) \) (w.r.t.\( x \)), we can legally antidifferentiate \( \text{deqtn2} \) with respect to \( x \) (on both sides) to get
\[ F(y(x)) = G(x) + C \]

which is what we got by magic before!
Example 1 page 32: We wish to solve

\[
\frac{dy}{dx} = -6xy
\]
\[
y(0) = 7
\]

Work:

Notice, our method for the general solution doesn’t actually give us the solution \(y(x)=0\). Solutions which exist to separable DE’s which are in addition to the ones we get are called "\textbf{singular solutions.}"

\textbf{slope field picture:}

\[
> \text{restart:with(plots):with(DEtools):}
> \text{deqtn:=diff(y(x),x)=-6*x*y(x): \ #this is example 2}
> \text{dsolve({deqtn,y(0)=7},y(x)); \ #Maple solution}
> \text{DEplot(deqtn,y(x),x=-2..2,[[y(0)=7],[y(0)=-4]],y=-10..10,arrows=line, color=black,linecolor=black,dirgrid=[30,30],stepsize=.1,}
> \text{title='Figure 1.4.1 page 31'}); \ #slope field with two solution graphs}
\]

\[
y(x) = 7 e^{-3x^2}
\]
Example extra: (#29 page 28)

\[ \frac{dy}{dx} = 3y^{(2/3)} \\
y(-1) = -1 \]

Solution: (There is a twist to this problem that will let us discuss the existence-uniqueness theorem.)

> deqtn:=diff(y(x),x)=3*abs(y(x))^(2/3.0)*sign(y(x)):
> DEplot(deqtn,y(x),x=-2..2,{[y(-1)=-1]},y=-5..5,arrows=line,
> color=black,linecolor=black,dirgrid=[30,30],stepsize=.1,
> title=`how many solutions really?`); #Maple missed a few
Examples 2-3 page 32

\[ \frac{dy}{dx} = \frac{4 - 2x}{3y^2 - 5}; \quad y(1) = 3 \]

Work:

\[ deqtn := \text{diff}(y(x), x) = \frac{4 - 2x}{3y(x)^2 - 5}; \]

\[ \text{part1} := \text{DEplot}(deqtn, y(x), x=-5..7, y=-5..5, \{y(1)=3\}, \text{arrows}=\text{line}, \]
\[ \quad \text{color}=\text{black}, \text{linecolor}=\text{black}, \text{dirgrid}=[40,40], \text{stepsize}=\text{.1}, \]
\[ \quad \text{title}=\text{`Part of Figure 1.4.2 page 32 `}); \]

\[ \text{with(plots)}; \]

\[ \text{part2} := \text{implicitplot}(y^3-5y=4x-x^2+9, x=-5..7, y=-5..5, \text{color}=\text{black}) ; \]

\[ \text{display} \{\text{part1, part2}\}; \]