A differential equation
\[ \frac{dy}{dx} = h(x, y) \]
is called **separable** iff \( h(x, y) \) is a product of a function of \( x \) times a function of \( y \),
\[ \frac{dy}{dx} = g(x) \phi(y) \]
This is equivalent to the DE
\[ \frac{dy}{dx} = \frac{g(x)}{f(y)} \]
where \( f \) and \( \phi \) are reciprocal functions.

**How to solve:**
The algorithm is very simple, but magic: treat \( \frac{dy}{dx} \) as a quotient of differentials (?!), and multiply through to rewrite the DE as
\[ f(y) \, dy = g(x) \, dx \]
Then antidifferentiate the left side with respect to \( y \) and the right side with respect to \( x \).
\[ \int f(y) \, dy = \int g(x) \, dx + C \]
If \( F(y) \) and \( G(x) \) are antiderivatives of \( f(y) \) and \( g(x) \), respectively, then this is the solution
\[ F(y) = G(x) + C \]
This equation defines \( y \) implicitly as a function of \( x \). Sometimes you can use algebra to explicitly solve for \( y \). The constant \( C \) can be adjusted to solve initial value problems.

**Why the method works:**
The use of differentials is disguising an application of the chain rule. Here is the explanation for the magic method: The differential equation
\[ \frac{dy}{dx} = \frac{g(x)}{f(y)} \]
CAN be rewritten as
\[ deqtn2 := f(y) \left[ \frac{dy}{dx} \right] = g(x) \]
If \( y(x) \) is any solution to \( deqtn2 \), then the left side, namely
\[ f(y(x)) \left[ \frac{dy}{dx} \right] \]
is the derivative with respect to \( x \) of
\[ F(y(x)) \]
whenever \( F(y) \) is an antiderivative of \( f(y) \) (with respect to \( y \)). This is just the chain rule! Thus if \( G(x) \) is any antiderivative of \( g(x) \) (w.r.t.\( x \)), we can legally antidifferentiate \( deqtn2 \) with respect to \( x \) (on both sides) to get
which is what we got by magic before!

**Example 1 page 31**: We wish to solve

\[ \frac{dy}{dx} = -6xy \]

\[ y(0) = 7 \]

Work:

Notice, our method for the general solution doesn’t actually give us the solution \( y(x) = 0 \). Solutions which exist to separable DE’s which are in addition to the ones we get are called "**singular solutions.**"
Example extra:

\[ \frac{dy}{dx} = 3y^{(2/3)} \]

\[ y(-1) = -1 \]

Solution: (There is a twist to this problem that will let us discuss the existence-uniqueness theorem.)

```maple
> deqtn := diff(y(x), x) = 3*abs(y(x))^(2/3.0)*sign(y(x));
DEplot(deqtn, y(x), x=-2.2, {[y(-1)=-1]}, y=-5..5,
    arrows=line, color=black, linecolor=black,
```
Example 3 page 32

\[ \frac{dy}{dx} = \frac{4 - 2x}{3y^2 - 5} \]
\[ y(1) = 3 \]

Work:
restart; with(DEtools); with(plots);
Warning, the name changecoords has been redefined

> deqtn := diff(y(x), x) = (4-2*x)/(3*y(x)^2-5):
DEplot(deqtn, y(x), x=-5..5, y=-5..5,
arrows=line, color=black, linecolor=black,
dirgrid=[40,40], stepsize=.1,
title=`Figure 1.4.2 page 32 `);