Recall geometric & algebraic ways of defining:
- Vectors
- Vector addition
- Scalar multiplication
- Magnitude

Finish Wed notes:
- Page 3-4: Vector algebra properties
  - Unit vectors
  - The standard basis vectors $\hat{i} \hat{j} \hat{k}$

§13.3 ($\mathbb{R}^2$), §14.2 ($\mathbb{R}^3$):

**Dot Product**: (Another algebraic operation on vectors with geometric meaning.)

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$
$\vec{b} = \langle b_1, b_2, b_3 \rangle$

Then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

"$\vec{a}$ dot $\vec{b}$"

(In $\mathbb{R}^n$, $\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i$)

1. $\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle =$
   $\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle =$

**Dot Product Algebra**:
1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ (so $|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$)
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
   $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(c \vec{a}) \cdot \vec{b} = c (\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c \vec{b})$
   $(c \text{ a scalar!})$
5. $\vec{a} \cdot \vec{0} = 0$

**2. Check some of these algebra rules**
   (to help you become comfortable using them)
The reason dot products are useful is that they let us do geometry. (As we will see.) The geometric meaning of dot product is contained in the formula:

**Theorem:** \[ \overrightarrow{a} \cdot \overrightarrow{b} = \lVert \overrightarrow{a} \rVert \lVert \overrightarrow{b} \rVert \cos \theta \]

(where \( \overrightarrow{a}, \overrightarrow{b} = \Theta \))

Thus

(i) \( \cos \theta = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{\lVert \overrightarrow{a} \rVert \lVert \overrightarrow{b} \rVert} \)

(ii) \( \overrightarrow{a} \perp \overrightarrow{b} \) exactly if \( \overrightarrow{a} \cdot \overrightarrow{b} = 0 \)

3. 14.1 #7) (which was HW due today), worked with dot product instead of distance formula:

Show \( P = (2,1,6) \)

\( Q = (4,7,9) \)

\( R = (8,5,-6) \)

are vertices of a right triangle.
Why the theorem on page 2 is true:
Consider the triangle

We will compute \( \| \mathbf{c} \|^2 \) two ways, set the expressions equal, and deduce \( \mathbf{a} \cdot \mathbf{b} = \| \mathbf{a} \| \| \mathbf{b} \| \cos \theta \) (after canceling terms)

(i) Law of cosines:
\[
\| \mathbf{c} \|^2 = \| \mathbf{a} \|^2 + \| \mathbf{b} \|^2 - 2 \| \mathbf{a} \| \| \mathbf{b} \| \cos \theta
\]
\[\text{--- generalizes Pythagorean Theorem, and it is actually a consequence of it, see HW.}\]

(ii) Vector computation:
\[
\mathbf{c} = \mathbf{b} - \mathbf{a}
\]
\[\text{(because, } \mathbf{a} + (\mathbf{b} - \mathbf{a}) = \mathbf{b} !)\]

So \( \| \mathbf{c} \|^2 = \mathbf{c} \cdot \mathbf{c} = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{b} \cdot \mathbf{b} - 2 \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} = \| \mathbf{a} \|^2 + \| \mathbf{b} \|^2 - 2 \mathbf{a} \cdot \mathbf{b} \]

Thus (setting computation (i) = computation (ii))
\[
\| \mathbf{a} \|^2 + \| \mathbf{b} \|^2 - 2 \| \mathbf{a} \| \| \mathbf{b} \| \cos \theta = \| \mathbf{a} \|^2 + \| \mathbf{b} \|^2 - 2 \mathbf{a} \cdot \mathbf{b}
\]
\[
-2 \| \mathbf{a} \| \| \mathbf{b} \| \cos \theta = -2 \mathbf{a} \cdot \mathbf{b}
\]

\[
\mathbf{a} \cdot \mathbf{b} = \| \mathbf{a} \| \| \mathbf{b} \| \cos \theta \]

On Wednesday (next class) we'll use
- dot product to understand
  1) the graph of \( ax + by + cz = d \) is a plane, with \( \perp \) vector \( \langle a, b, c \rangle \)
  2) how to project one vector orthogonally onto another, using only algebra

\[
\text{proj}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{b} \cdot \mathbf{b}}{\| \mathbf{a} \|^2} \mathbf{a}
\]

3) Compute "work" when moving against constant force fields.
Homework for Friday 1/21: As always, circled problems hand in, others recommended.

1. Verify the law of cosines
   \[ \ell_1^2 = \ell_2^2 + \ell_3^2 - 2 \ell_2 \ell_3 \cos \theta \]
   using trig, as indicated.

2. \( \theta = 0, \pi/2, \pi \). (These are special cases; \( \theta = \pi/2 \) is just Pyth, so you don't need to redo it, after last HW)

3. \( 0 < \theta < \pi/2 \). Label so that \( \ell_1 \leq \ell_2 \). Find lengths \( \ell_1, \ell_2 \) from trig.
   Then apply Pyth. thm.

4. \( \pi/2 < \theta < \pi \). Follow same steps as in 3:

Book problems

13.2 2, 3, 4, 8, 9, 12, 17, 19
13.3 1, 3, 5, 6, 7, 9, 14, 15, 16, 23, 27, 28, 30, 33a
14.2 1a (draw box with pts at opposite corners, to help see.)
32, 3, 7, 10, 11, 12, 13, 15, 18, 23, 25a, 26, 39, 31, 33, 40, 43, 45, 50