9/12.2 curvature & acceleration

\[ \text{curvature is a quantitative measure of how "fast" a curve is bending.} \]

\[ \text{w.r.t.?} \]

\[ \text{how about, if you traverse the curve with unit speed (|v(t)| = 1),} \]

\[ \text{digress:} \]

\[ \text{unit speed re parameterizations:} \]

\[ \text{example: } F(t) = \begin{bmatrix} 10 \cos t \\ 10 \sin t \end{bmatrix}, \quad 0 \leq t \leq 2\pi \]

\[ \text{parameterizes the circle } x^2 + y^2 = 100 \]

\[ \text{of radius 10 centered at the origin} \]

\[ F'(t) = \begin{bmatrix} -10 \sin t \\ 10 \cos t \end{bmatrix} \]

\[ |F'(t)| = 10, \text{ so we're going 10 times too fast} \]

\[ \text{If } s = 10t; \quad t = \frac{s}{10} \]

\[ \text{then } \tilde{F}(s) = F(t(s)) = \begin{bmatrix} 10 \cos \frac{s}{10} \\ 10 \sin \frac{s}{10} \end{bmatrix} \]

\[ \text{satisfies } |\tilde{F}'(s)| = 1! \]

\[ \text{In general, if } F: [a,b] \to \mathbb{R}^n \]

\[ \text{Define } s(t) = \int_a^t |F'(t)| \, dt = \text{length of arc of curve between } F(a) \text{ and } F(t) \]

\[ \text{1210 FTC } \Rightarrow \frac{ds}{dt} = |F'(t)| = v(t) \]

\[ \text{(makes sense: how fast the arc length is changing equals speed)} \]

\[ \text{Assuming } v(t) > 0 \text{ on } [a,b], \text{ s(t) is a strictly increasing function, so } s(a) = 0 \]

\[ s(b) = L \text{ total length} \]

\[ \text{there is an inverse function } t(s), \quad t: [0,L] \to [a,b], \]

\[ \text{and } \tilde{F}(s) := F(t(s)) \text{ is unit speed, since } \tilde{F}'(s) = \frac{dF}{dt} \frac{dt}{ds} = \frac{v}{v} = 1.! \]
A unit speed parameterization $\tilde{r}(s)$ is also called an arclength parameterization, since then
$$\int_0^S \| \tilde{r}'(t) \| dt = \int_0^S 1 = S,$$
so the length of the curve as $t$ varies from 0 to $S$ exactly equals $S$.

For a plane curve:

\[ \frac{\overrightarrow{T}(s)}{\kappa} = \text{unit tangent vector at } \overrightarrow{r}(t) = \overrightarrow{r}(t(s)), \]

\[ \kappa := \left| \frac{d\phi}{ds} \right| \]

Notice, \[ \overrightarrow{T}(s) = \begin{bmatrix} \cos \phi(s) \\ \sin \phi(s) \end{bmatrix} \]
\[ \overrightarrow{T}'(s) = \begin{bmatrix} -\sin \phi(s) \\ \cos \phi(s) \end{bmatrix} \]
so \[ |\overrightarrow{T}'(s)| = |\phi'(s)| = \kappa \]

Example:
\[ \overrightarrow{r}(s) = \begin{bmatrix} 10 \cos \frac{s}{10} \\ 10 \sin \frac{s}{10} \end{bmatrix} \]
\[ \overrightarrow{r}'(s) = \frac{\overrightarrow{T}(s)}{\kappa} = \begin{bmatrix} -\sin \frac{s}{10} \\ \cos \frac{s}{10} \end{bmatrix} \]
\[ \overrightarrow{r}''(s) = \begin{bmatrix} -\frac{1}{10} \cos \frac{s}{10} \\ \frac{1}{10} \sin \frac{s}{10} \end{bmatrix} \]
\[ |\overrightarrow{T}'(s)| = \kappa = \frac{1}{10} \]

So, what is the curvature $\kappa$ of a radius $R$ circle?
Computing curvature for plane curves:

You don't need to find the explicit arc length reparameterization, you just need to be good at the chain rule.

\[
\begin{align*}
\vec{F}(t) & = [x'(t), y'(t)]^T \\
\phi & = \arctan \frac{y'(t)}{x'(t)} \\
T(t) & = \frac{1}{\sqrt{v^2}} F'(t), \quad v = \sqrt{\dot{x}'^2 + \dot{y}'^2} \\
\frac{d\phi}{ds} & = \frac{d\phi}{dt} \frac{dt}{ds} = \frac{1}{v} \frac{d\phi}{dt} \\
& = \frac{1}{v} \left( \frac{1}{1 + (\frac{y'}{x'})^2} \right) \frac{dt}{dx} (\frac{y'}{x'}) \\
& = \frac{1}{v} \frac{y''x' - y'x''}{(x')^2} \\
& = \frac{y''x' - y'x''}{k(x'^2 + y'^2)^{3/2}} \\
\end{align*}
\]

So \( k = \frac{|y''x' - y'x''|}{(x'^2 + y'^2)^{3/2}} \)

\text{Example: Find the curvature of the parabola } y = x^2, \text{ at every point on the parabola.}

Let \( \vec{F}(t) = [t, t^2]^T \)

\[
\begin{align*}
x & = t, \quad x' = 1, \quad x'' = 0 \\
y & = t^2, \quad y' = 2t, \quad y'' = 2 \\
k & = \frac{12t - 2t^3}{(1 + 4t^2)^{3/2}} = \frac{2}{(1 + 4t^2)^{3/2}} \\
\end{align*}
\]

So at \([x, x^2]^T\), \( k = \frac{2}{(1 + 4x^2)^{3/2}} \)

\text{e.g. at } x = 0, \quad k = 2

\text{The osculating circle has same curvature as curve, is tangent to curve (all at a fixed point), and on the side of the curve which properly reflects the curve's bending.}
Space curves \( \mathbf{F} : [a,b] \to \mathbb{R}^n \) any \( n \geq 2 \)

We can't use \( \phi \) when \( n \neq 2 \)
but the definition
\[
\kappa = \left| \frac{d\mathbf{T}}{ds} \right|
\]

still holds.

Notice, since \( \mathbf{T} \cdot \mathbf{T} = 1 \)
\[
\frac{d}{ds} \mathbf{T} \cdot \mathbf{T} = 0
\]
\[
2 \frac{d^2\mathbf{T}}{ds^2} \cdot \mathbf{T} = 0
\]

So, if \( \frac{d^2\mathbf{T}}{ds^2} \neq 0 \) we call \( \mathbf{N} := \frac{1}{\left| \frac{d\mathbf{T}}{ds} \right|} \frac{d\mathbf{T}}{ds} \) the unit normal to the curve.

Compactively
\[
\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt} \frac{dt}{ds}
\]
\[
= \frac{1}{v} \frac{d\mathbf{T}}{dt}
\]
\[
= \frac{1}{v} \frac{d}{dt} \left( \frac{1}{v} \mathbf{F}(t) \right)
\]

So
\[
\kappa = \frac{1}{v} \left| \frac{d}{dt} \frac{1}{v} \mathbf{F}(t) \right|
\]

Back to physics:
Normal and tangential components of acceleration

\[
\mathbf{T}(t) = \frac{\mathbf{F}'(t)}{v}
\]
so
\[
\mathbf{F}'(t) = v \mathbf{T}(t)
\]
\[
\mathbf{F}''(t) = v' \mathbf{T}(t) + v \frac{d}{dt} \mathbf{T}(t)
\]
\[
= v' \mathbf{T}(t) + \kappa v^2 \mathbf{N}
\]

For plane curves you can get \( \mathbf{N} \) directly from \( \mathbf{T} \)

E.g. in parabola example at [1].