Math 2210-1
Fri Oct 29.

Multiple Integration b 14.1
Double integrals

HW for Wed Nov. 3

14.1 (11) (12) ← also, in 11, 12 do a Riemann
sum approximation for each integral,
using unit length squares and
evaluating \( f \) at the midpoints.

\[
\begin{align*}
(11) & \quad (2) \\
(12) & \quad (3, 0) \\
(2) & \quad (4) \\
\end{align*}
\]

14.1 cont'd 27, 31, 32
14.2 3, 7, 8, 15, 18, 23, 25, 29, 31

Remember 1210 integral:
\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i) \Delta x_i
\]

we motivated this definition by thinking about area under a graph \( y = f(x) \), but the definite integral had lots of other applications:

- area between graphs
- total mass, if \( f(x) = \text{mass/length} \)
- work, if \( f(x) = \text{force} \)
- curve length
- volumes (e.g., revolution)
- surface area of revolution
- centers of mass

If \( f \) is continuous on \([a, b]\), then the definite integral exists.

- Fundamental Theorem of Calc says you can compute
  \[
  \int_a^b f(x) \, dx = F(b) - F(a)
  \]
  where \( F \) is any antideriv. of \( f \).
Double integrals over rectangles

Let \( R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 \text{ s.t. } a \leq x \leq b, c \leq y \leq d\} \) be a coordinate rectangle.

Let \( z = f(x,y) \) a function with domain \( R \).

Partition \( P \) of rectangle.

\[
a = x_0 < x_1 < \cdots < x_m = b
\]

\[
c = y_0 < y_1 < \cdots < y_n = d.
\]

![Partition diagram]

This partition \( P \) yields subrectangles

\[
R_{ij} = [x_{i-1},x_i] \times [y_{j-1},y_j], \quad \text{with area } \Delta A_{ij} = (x_i-x_{i-1})(y_j-y_{j-1})
\]

Pick points \((x^*_i, y^*_j)\) in \( R_{ij} \).

Then, for this partition \( P \), and function \( f \), get

Riemann Sum

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_i, y^*_j) \Delta A_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_i, y^*_j) (x_i-x_{i-1})(y_j-y_{j-1})
\]

This Riemann sum can be thought of as an approximation of the volume between \( z = f(x,y) \) and the \( x-y \) plane, if \( f \) is positive.

Example. (to do!)

\[f(x,y) = x+y\]
\[R = [0,2] \times [0,2].\]

\( P: \)

\[
\begin{array}{ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}
\]

Find the Riemann sum using midpoints (\( *'s \)).

Theorem. If \( f \) is continuous on the rectangle \( R \), then

\[
\lim_{\Delta P \to 0} \text{(Riemann sum of } f) = \iint_R f(x,y) \, dA
\]

exists. (Called the double integral of \( f \) over the region \( R \).)

Luckily, it is easy to compute...
**Theorem:** Double integrals as iterated single integrals, (for $f$ continuous on rectangle $[a,b] \times [c,d]$)

$$
\iint_R f(x, y) \, dA = \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx = \int_c^d \left( \int_a^b f(x, y) \, dx \right) \, dy
$$

Idea of why: consider Riemann sum

$$
\sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{i,j}^*, y_{j}^*) \Delta x \Delta y
$$

if we let max $\Delta x \to 0$, keeping $y_j^*, \Delta y_j$ fixed, 

this should converge to

$$
\sum_{j=1}^{n} \int_a^b f(x, y_j^*) \Delta y_j = \sum_{j=1}^{n} \left( \int_a^b f(x, y^*_j) \, dx \right) \Delta y_j
$$

which is a Riemann sum for the function $G(y) = \int_a^b f(x, y) \, dx$, $c \leq y \leq d$

So, as $\Delta y_j \to 0$, this Riemann sum should converge to

$$
\int_c^d G(y) \, dy = \int_c^d \left( \int_a^b f(x, y) \, dx \right) \, dy
$$

(get reverse iteration analogously).

Another way to "see" this is to recall how you found volumes by slicing, in Calculus.

$$
A(y) = \text{area when } y-\text{cord is } y = \int_a^b f(x, y) \, dx
$$

$$
\text{Vol} = \iiint f(x, y, z) \, dV = \int_c^d A(y) \, dy = \int_c^d \left( \int_a^b f(x, y) \, dx \right) \, dy
$$
Example \( R = [0, 2] \times [0, 1] \)

\[ f(x, y) = xy. \]

\[
\iint_R (x+y) \, dA = \int_0^2 \left( \int_0^2 x+y \, dy \right) \, dx = \int_0^2 \left[ xy + \frac{y^2}{2} \right]_0^2 \, dx = \int_0^2 2x + 2 \, dx = [x^2 + 2x]_0^2 = 8
\]

How does this compare to the Riemann sum on page 2? Anecdote?

Example \( R = [0, \pi/2] \times [0, 1] \)

\[ f(x, y) = e^x + \cos x. \]

Compute \( \iint_R f(x, y) \, dA \) as iterated integrals, in both orders.