Max-min problems by 13.9

Remember in Calc.

If $f$ is continuous on a closed interval $[a,b]$.

Then $f$ attains its extreme values, (max & min), either

(i) at the endpoints (the "boundary" of the interval)

or

(ii) at critical points $x_0$ in $(a,b)$ (in the "interior" of the interval)

\[
\Rightarrow \text{ either } f'(x_0) = 0 \text{ or } f'(x) \text{ DNE.} \quad \text{[Because, if } f'(x_0) \text{ exists and } \not= 0, \text{ then } f \text{ is inc or dec at } x_0, \text{ so can't have even a local extremum.]}\]

The analogous fact is true in n-dimensional domain

$\mathbf{f} : D \rightarrow \mathbb{R}^n$

If $D$ is a closed domain

$\Downarrow$

$D$ includes its boundary.

and if $D$ is a bounded domain

$\Rightarrow$ all points in $D$ are at most a fixed finite distance from $\partial D$.

and if $f$ is continuous on $D$

Then $f$ attains its extreme values, either

(i) along the boundary of $D$

or

(ii) at critical points $x_0$ in $D$

\[
\Rightarrow \text{ either each } \frac{\partial f}{\partial x_i}(x_0) = 0 \text{ at } x_0
\]

or not all partial derivatives exist at $x_0$.

The reasoning is just like in Calc : if some $\frac{\partial f}{\partial x_i}(x_0) \not= 0$

then $f$ is increasing or decreasing in the $x_i$ direction, at $x_0$. 
Example

\( \text{Let } D \text{ = rectangle, } -4 \leq x \leq 4 \)
\( -6 \leq y \leq 6 \)

\[ F(x, y) = \frac{3}{4} y^2 + \frac{1}{2} y^3 - \frac{1}{32} y^4 - x^2 \]

Find the extreme values of \( F \), by finding
all critical points and analyzing the behavior
of \( F \) along the boundary (perimeter) of the rectangle.

\[ n \quad y = 6, \quad F(x, 6) = -4.5 - x^2 \]

\[ \begin{array}{c}
\text{domain} \\
F(x, -6) = -22.5 - x^2 \\
\end{array} \]

Critical points:

\[ F_x = -2x = 0 \rightarrow \text{so } x = 0 \]
\[ F_y = \frac{3}{2} y + \frac{1}{8} y^2 - \frac{1}{8} y^3 = 0 \]
\[ = -\frac{1}{8} \left[ y^3 - y^2 - 12y \right] \]
\[ = -\frac{1}{8} y (y^2 - y - 12) \]
\[ = -\frac{1}{8} y (y-4)(y+3) \rightarrow \text{so } y = 0, 4, -3 \]

\[ (0, 0), (0, 4), (0, -3) \]

Looking along the boundary:

we can approach this
systematically, and it
helps to use technology
(next page)

\[ F(0, 0) = 0 \]
\[ F(0, 4) = 6 \frac{3}{4} \]
\[ F(0, -3) \approx 3.09 \]
> with(plots):
Warning: the name changecoords has been redefined

> F := (x, y) -> .75*y^2 + 1/24*y^3 - 1/32*y^4 - x^2;
> plot3d(F(x, y), x = -4..4, y = -6..6,
axes = boxed, color = white);

\[ f : = y \rightarrow 0.75y^2 + \frac{1}{24}y^3 - \frac{1}{32}y^4 \]

\[ 0, 4, 6 \]

> f(4); f(-3); f(0); f(6); f(-6):
6.66666667
3.093750000
0.0
-4.50000000
-22.50000000

> F(4, -6); F(-4, -6); # minimum values on the rectangle
F(0, 4); # maximum value on the rectangle
-38.50000000
-38.50000000
6.66666667

>
open-topped box.
Volume to be $4 \text{ m}^3$
What dim's $x, y, z$ minimize surface area (cardboard area)?
Example.

What line minimizes the "squared vertical deviations" for points $[0], [1], [2]$

\[ y = mx + b \]

\[ F(m, b) = (b - 1)^2 + (m + b - 2)^2 + (2m + b - 1)^2 \]
Homework for Wed 10/20

5 13.4 7 9,13, 22, 32, 38, 41, 43, 45, 50, 55, 56, 57, 59
5 13.5 9 11, 14, 15, 25, 36, 37, 39, 52, 59, 61
5 13.6 17 24, 29, 36, 38

And

(1) page 991 #51 (linear regression in general).
   (a) Show that \( m \) and \( b \) satisfy the matrix eqtn
   \[
   \begin{bmatrix}
   \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i \\
   \sum_{i=1}^{n} x_i & n
   \end{bmatrix}
   \begin{bmatrix}
   \hat{m} \\
   \hat{b}
   \end{bmatrix}
   =
   \begin{bmatrix}
   \sum_{i=1}^{n} x_i y_i \\
   \sum_{i=1}^{n} x_i y_i
   \end{bmatrix}
   \]

   (b) invert the 2x2 matrix above to find \( m \) and \( b \).
   (c) verify that on class example (page 5) values are reproduced by this general formula.

(II) Consider the lines
   \[\bar{r}(t) = t \begin{bmatrix} 0 \\ 0 \end{bmatrix}\]
   \[\bar{e}(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\]

Find the nearest the lines approach each other by minimizing
   \[|\bar{r}(t) - \bar{e}(s)|^2.\]