Math 2210-1
Monday 11/8
§14.5
Moments, centers of mass, volume
Applications of double integrals:

**Mass**

- **Discrete**
  
  \[
  \text{total mass} = \sum_i m_i
  \]

**Centers of mass**

On a line:

\[
\bar{x} = \frac{\sum m_i x_i}{\sum m_i}
\]

\(\bar{x}\) satisfies no net torque condition:

\[
\sum i m_i (x_i - \bar{x}) = 0
\]

\(\sum m_i x_i = \bar{x} \sum m_i = 0\)

Masses in a plane have center of mass \((\bar{x}, \bar{y})\)

\[
\sum m_i (x_i - \bar{x}) = 0
\]

\[
\sum m_i (y_i - \bar{y}) = 0
\]

So,

\[
\bar{x} = \frac{\sum m_i x_i}{m} = \frac{M_{\text{mass}}}{m}
\]

\[
\bar{y} = \frac{\sum m_i y_i}{m} = \frac{M_{\text{mass}}}{m}
\]

Lamina

\[
m = \iint_R \delta(x,y) \, dA
\]

where \(\delta(x,y)\) is the density function.

\[
m = \iint_R \delta(x,y) \, dA
\]

\[
\bar{x} = \frac{\iint_R (x - \bar{x}) \delta(x,y) \, dA}{m}
\]

\[
\bar{y} = \frac{\iint_R (y - \bar{y}) \delta(x,y) \, dA}{m}
\]

\[
\bar{x} = \frac{\iint_R x \delta(x,y) \, dA}{m}
\]

\[
\bar{y} = \frac{\iint_R y \delta(x,y) \, dA}{m}
\]
Remarks: If the density is constant in a lamina, then this constant can be factored out of the numerator & denominator of the formulas for $\bar{x}$ & $\bar{y}$, so we may as well assume $\delta = 1$ to compute $\bar{x}, \bar{y}$ in this case.

Example: Find the centroid of the uniform half disk

\[ x^2 + y^2 \leq a^2 \]
\[ y \geq 0 \]

- Explain why $\bar{x} = 0$

- Find $\bar{y}$
Volumes of revolution

Pappus' theorem: Let $R$ be a region, $L$ be an axis
in the plane
on one side of this region.

Consider the volume of revolution
obtained by rotating $R$ about $L$.

Then volume $V = (2\pi r) A$

where $A$ is area of region $R$
and $r$ is distance from axis to centroid $\left[ \frac{\bar{z}}{g} \right]$.

Example: Use page 2 example
to verify the centroid computation
you did there is consistent with Pappus,
since volume of radius $a$-ball is $\frac{4}{3} \pi a^3$.

Proof of Pappus I

Assume $y$-axis is
axis of rotation.

and $x \geq 0$ for $R$ points.

d$V = (2\pi x) dA$

$V = \iint 2\pi x dA$

$= 2\pi \int \int x dA$

$= \bar{x} A$.\n
Example: What is the volume of this torus (doughnut)?

\[(x-A)^2 + y^2 \leq a^2\]

obtained by rotating the disk about the y-axis.

There are other applications in this section: moments of inertia, 2nd Poppi, then which we won't cover in class (or have exams).