§ 15.2-15.3 Curve integrals

\[ \int_a^b f(t) \, \frac{d\vec{r}}{dt} \, dt \]

\[ \int_C \vec{F}(t) \cdot d\vec{s} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \]

Last Wednesday we did examples of curve integrals of type

\[ \int_C f(x) \, ds := \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| \, dt \]

If \( f \) is a scalar function, e.g., to find mass or moments of a wire.

Today: "Line integrals"

Let \( \vec{F}(x) \) be a vector field defined near the curve \( C \).

Then

\[ \int_C \vec{F} \cdot d\vec{s} := \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \]

The book writes

\[ d\vec{s} = |\vec{v}(t)| \, dt \]

If \( \vec{F} = \langle P, Q, R \rangle \)

This is also written as

\[ \int_C P \, dx + Q \, dy + R \, dz \]

Physics connection: \( \int_C \vec{F} \cdot d\vec{s} \) is defined to be the work \( W \) done by the vector field \( \vec{F} \) in moving a particle along \( C \).

(This is defined to be the opposite of the work done by the particle.)
Example 1 Consider the inverse square force field $\mathbf{F}$ attraction to the origin:  
\[
\mathbf{F}(x, y, z) = -\frac{C}{r^3} = -\frac{C}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{r}. \tag{c>0}
\]

Let $C$ be the line segment from $[\frac{8}{3}, \frac{4}{3}]$ to $[\frac{8}{3}, 0]$. 

Find the work done by the field in moving a particle along $C$. 

1. Take $F(t) = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$, $0 \leq t \leq 4$. 
\[\text{i.e.} \quad x = t, \quad y = 3, \quad z = 0.\]
\[
\mathbf{F}(t) = \frac{-C}{(t^2 + 9)^{3/2}} \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{F}'(t) = \frac{-Ct}{(t^2 + 9)^{3/2}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{F}(t) \cdot \mathbf{F}'(t) = \frac{-Ct}{(t^2 + 9)^{3/2}}.
\]
\[
\int_C \mathbf{F} \cdot d\mathbf{x} = \int_0^4 \frac{-Ct}{(t^2 + 9)^{3/2}} \, dt = C \left[ -\frac{1}{2} \left( \frac{9}{2} \right)^{1/2} \right]_0^4 = C \left[ -\frac{1}{2} \left( \frac{9}{2} \right)^{1/2} \right] = -\frac{2C}{15},
\]

Notice work done by field is negative 

(so work done by object is positive). 

2. "Same" computation. 

Use $x$ as the parameter. 
\[
\mathbf{F}(x) = \begin{bmatrix} x \\ 3 \\ 0 \end{bmatrix}, \quad \int_C P \, dx + Q \, dy + R \, dz = \int_{x=0}^4 \frac{-C}{(x^2 + 9)^{3/2}} \, dx = \text{same answer.}
\]

\[
dy = dz = 0.
\]

Does it matter how you parameterize a curve? 

So reparameterizations on the curve give same value for \( \int_C \mathbf{F} \cdot d\mathbf{x} \) 

\[
\int_C \mathbf{F} \cdot d\mathbf{x} = \lim_{n \to \infty} \sum_{i=1}^{n} \mathbf{F}(x_i) \cdot \Delta \mathbf{x}_i,
\]

So, if you reverse direction you get the opposite value for this integral. 

\[
\int_C \mathbf{F} \cdot d\mathbf{x} = \lim_{n \to \infty} \sum_{i=1}^{n} \mathbf{F}(x_i) \cdot \Delta \mathbf{x}_i,
\]

Length preserving reparamters yield same value.
Example 2

\( \vec{F}(x,y) = \langle x + y, -x \rangle. \)

\( C_1 = \) parabola \( y = x^2 \) from \((0,0)\) to \((1,1)\)

\( C_2 = \) line segment from \((0,0)\) to \((1,0)\)

\( C_3 = \) triangle-by line segment from \((1,0)\) to \((1,1)\).

\[
\text{compute} \int_{C_1} \vec{F} \cdot \, dx = \int_{C_1} (2x+y) \, dx - x \, dy
\]

\( \text{ans} = \frac{2}{3}. \)

\[
\text{compute} \int_{C_2 \cup C_3} \vec{F} \cdot \, dx
\]

\( \text{book might write} \quad C_2 + C_3 \)

\( \text{ans} = 0. \)
Line integrals \( \int_C \vec{F} \cdot d\vec{x} \) are called \textit{path independent} if the values only depend on the starting and final point of the curve, not the route taken.

**Theorem** If \( \vec{F} \) is a gradient field, \( \vec{F} = \nabla f \), then

\[
\int_C \vec{F} \cdot d\vec{x} = f(B) - f(A) \quad \text{where } A \text{ is starting point and } B \text{ is final point.}
\]

[In fact, every vector field which gives rise to path independent line integrals is a gradient field!]

**Proof.** Let \( \vec{F} : [a, b] \rightarrow \mathbb{R}^n \)

- parametrize \( C \), \( \vec{F}(a) = A \) \& \( \vec{F}(b) = B \).

\[
\int_C \vec{F} \cdot d\vec{x} := \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt
\]

\[
= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \, dt
\]

\[
= \frac{d}{dt} f(\vec{r}(t)) \quad \text{by multivar. chain rule!}
\]

\[
= f(\vec{r}(b)) - f(\vec{r}(a)) \quad \text{by FTC (Fundamental Theorem of Calculus)}
\]

\[
= f(B) - f(A).
\]

**Example 1** page 2

- examples:
  - \( \vec{F} = -\frac{\vec{r}}{||\vec{r}||^3} \) is a gradient field,
  - \( \vec{F} = \nabla \left( \frac{x}{\sqrt{x^2+y^2+z^2}} \right) \).

\[
\int_{s_0}^{s_1} \vec{F} \cdot d\vec{x}
\]

\[
= f(B) - f(A)
\]

\[
= c \left( \frac{1}{2} - \frac{1}{3} \right)
\]

**Example 2** cannot be a gradient field, by our computations on page 3.

Also, \( f_x = (2x+y) \) \{ not possible! \}.

\( f_y = -x \)