1. The Figure 3.4.1 (page 182) shows the motion of a mass attached to a spring.
   (i) Assuming that there is no external force, derive the following second-order linear differential equation
   \[ mx'' + cx' + kx = 0. \]
   
   (ii) Find the general solution if \( c^2 > 4km \)

   (iii) Find the general solution if \( c \neq 0 \) and \( c < 4km \).

   (iv) If there exists an external force \( F \), then we get
   \[ mx'' + cx' + kx = F. \]
   Assume \( m = 1, c = 0, k = 9, F = 80\cos 5t \). Find \( x(t) \) if \( x(0) = x'(0) = 0 \).
2.

(i) Derive the equation

\[ m_1 x'' = -(k_1 + k_2)x + k_2 y, \]
\[ m_2 y'' = -k_2(y - x) + f \]

for the displacements from equilibrium of the two masses shown in Figure 4.1.1 (page 243).

(ii) Assume \( m_1 = 2, m_2 = 1, k_1 = 4, k_2 = 2 \) and \( f = 40 \sin 4t \). Solve \( x(t) \) and \( y(t) \).
3. Consider three brine tanks containing \( V_1 = 40, V_2 = 40 \) and \( V_3 = 40 \) gallons of brine, respectively. Fresh water flows into tank 1, while mixed brine flows from tank 1 into tank 2, from tank 2 into tank 3, and out of tank 3. Let \( x_i(t) \) denote the amount of salt in tank \( i \) and assume that each flow rate is \( r = 10 \) gallons per minute. Find \( x_i(t) \) such that

\[
x_1(0) = 10, \quad x_2(0) = x_3(0) = 0.
\]
4. Find the solution of the system

\[ \mathbf{X}' = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix} \mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \]
5. Find the solution of the system

\[
X' = \begin{pmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}
\]

Hint: \( \lambda = -1, -1, -1 \).
6. Find the general solution of the equation

\[ y^{(4)} - 2y'' + y = x \cos x \]
7. Let $A$ be a $n \times n$ matrix with real entries. Let $\lambda$ be an Eigenvalue of $A$ and $v_1$ be an Eigenvector associated with $\lambda$.

(i) Prove the $X_1(t) = v_1 e^{\lambda t}$ is a solution of the system

$$X' = AX$$

(ii) Assume that $\lambda$ has multiplicity 2, $v_1$ is the only (independent) Eigenvector and $v_2$ is a non-zero vector such that

$$(A - \lambda I)v_2 = v_1.$$ 

Prove that $X_1$ and $X_2 = v_1 te^{\lambda t} + v_2 e^{\lambda t}$ are independent solutions of system (1).

(iii) Assume $\lambda = (i\alpha)^2 = -\alpha^2 < 0$. Prove that

$$v_1 \cos \alpha t, \quad v_1 \sin \alpha t$$

are two independent solutions of the system

$$X'' = AX$$