Exercises for Module 1

1. A monomer (represented by $A_1$) polymerizes to form polymer of length $n$, (denoted $A_n$) via the reaction scheme

$$A_n + A_1 \rightleftharpoons A_{n+1}$$

(a) Use the law of mass action to write a system of differential equations for the dynamics of $A_n$. What is the equation governing the dynamics of $A_1$? Check to be sure that $\sum_n n \frac{dA_n}{dt} = 0$.

(b) Assuming that $\sum_n n A_n = A_0$, find the steady state distribution of polymer lengths.

2. Use the law of mass action to find differential equations governing $S_{jk}$ for the reaction scheme

Use that $S_{00} + S_{01} + S_{10} + S_{11} = 1$.

(a) Suppose the “top” and “bottom” reactions are independent from the “left” and “right” reactions. Assume that $S_{10} = mh$, $S_{00} = (1-m)h$, $S_{11} = m(1-h)$, $S_{01} = (1-m)(1-h)$. Find the differential equations governing the dynamics of $m$ and $h$.

(b) Assume that the top and bottom reactions are fast compared to the left and right reactions. Use the quasi-steady state assumption to find $S_{01}$ in terms of $h = S_{00} + S_{01}$ and find the equation governing the dynamics of $h$. (This problem is the most difficult in this set. For help, look at Keener and Sneyd, Mathematical Physiology.)

3. Sketch the phase portrait for the system of differential equations

$$\frac{d\phi}{dt} = A\phi(1-\phi)(\phi - a) - h + I_0, \quad \frac{dh}{dt} = \epsilon(\phi - \gamma h)$$

where all parameters are positive, $0 < a < \frac{1}{2}$ and $\epsilon << 1$. What qualitatively different kinds of phase portraits are possible (there are 3)?

Write a simple Matlab code to simulate the solution of this equation (Use $A = 10$, $a = 0.1$, $\epsilon = 0.1$ for starters). Find parameter values for each of the qualitatively different behaviors.
1 Solutions

1. (a) The differential equations are

\[
\frac{dA_n}{dt} = k_+A_{n-1}A_1 - k_+A_nA_1 + k_-A_{n+1} - k_-A_n
\]  

(3)

for \( n \geq 2 \) and

\[
\frac{dA_1}{dt} = -2k_+A_1^2 + 2k_-A_2 + \sum_{n=3}^{\infty} k_-A_n - \sum_{n=2}^{\infty} k_+A_nA_1
\]  

(4)

To check this, note that

\[
\sum_{n=2}^{\infty} n \frac{dA_n}{dt} = \sum_{n=2}^{\infty} n(k_+A_{n-1}A_1 - k_+A_nA_1 + k_-A_{n+1} - k_-A_n)
\]

\[
= \sum_{n=2}^{\infty} nk_+A_{n-1}A_1 - \sum_{n=2}^{\infty} nk_+A_nA_1 + \sum_{n=2}^{\infty} nk_-A_{n+1} - \sum_{n=2}^{\infty} nk_-A_n
\]

\[
= \sum_{n=1}^{\infty} (n+1)k_+A_nA_1 - \sum_{n=2}^{\infty} nk_+A_nA_1 + \sum_{n=3}^{\infty} (n-1)k_-A_n - \sum_{n=2}^{\infty} nk_-A_n
\]

\[
= 2k_+A_1^2 + \sum_{n=2}^{\infty} k_+A_nA_1 - \sum_{n=3}^{\infty} k_-A_n - 2k_-A_2
\]

(5)

so that \( \sum_{n=1}^{\infty} n \frac{dA_n}{dt} = 0 \).

(b) To find the steady state solution, notice that the equation (3) in steady state \( \frac{dA_n}{dt} = 0 \) is a linear difference equation (with \( A_1 \) fixed). Therefore, for \( n \geq 2 \), \( A_n = \alpha \mu^n \), where

\[
k_+A_1 - k_- - \mu(k_+A_1 - k_-) = 0
\]

(6)

so that \( \mu = \frac{k_+A_1}{k_-} \). Notice that for consistency, \( A_1 = \alpha \mu \) so that \( \alpha = \frac{k_-}{k_+} \).

Now to find \( A_1 \) we only need to solve the equation \( \sum_{n=2}^{\infty} A_n + A_1 = A_0 \). However,

\[
\sum_{n=2}^{\infty} nA_n = \alpha \sum_{n=2}^{\infty} n\mu^n
\]

\[
= \alpha \sum_{n=2}^{\infty} n\mu^n = \alpha \left( \frac{1}{(1-\mu)^2} - \mu \right)
\]

(8)

This leaves us with a single equation for \( \mu \)

\[
\frac{k_-}{k_+} \left( \frac{1}{(1-\mu)^2} - \mu \right) = A_0.
\]

(9)