1 Regular Perturbation Theory

(due Sept. 22, 2015)

1. Find a two-term asymptotic expansion, for small $\epsilon$, for all solutions of the following equations:

(a) \[ \epsilon x^3 - 3x + 1 = 0 \]  

(b) \[ x^{2+\epsilon} = \frac{1}{x + 2\epsilon} \]  

(c) \[ xe^{-x} = \epsilon \]  

(d) \[ x^3 - x^2 - \epsilon = 0 \]

2. Find a two-term asymptotic expansion, for small $\epsilon$, of the solution of the following boundary value problems:

(a) \[ y'' + y + \epsilon y^3 = 0, \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 1. \] \( y'' - y = 0, \quad y(0) = 0, \quad y(1 + \epsilon) = 1 \) \( \text{Hint: Use a change of variables to transform this to a problem on a fixed domain (independent of $\epsilon$).} \)

3. Suppose that a one-dimensional cell that is 10 $\mu$m long and the voltage potential is held fixed at both ends (so that $\Phi = 0$ at the ends). Suppose that the cell contains approximately equal amounts (say 400mM) of sodium and chloride, but suppose there is a slight excess of sodium. How much excess sodium is there if the total difference between maximal and minimal potential in the cell is 0.1mV? Plot the (approximate) distribution of potential and ion concentrations for both species.

Hint: Derive the Poisson-Boltzman equation as follows: According to the Poisson equation,

\[ \epsilon \nabla^2 \phi = -qN_a \sum_i z_i c_i \]  

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where \( \epsilon \) is the dielectric constant of the medium, \( N_a \) is Avagadro's number, \( q \) is the unit electric charge, \( c_i \) and \( z_i \) are the concentration and unit charge for the \( i^{th} \) species. According to the Nernst-Planck equation,

\[
0 = \nabla c_i + \frac{z_i F}{RT} c_i \nabla \phi. \tag{8}
\]

Use this information to derive the Poisson-Boltzmann equation in dimensionless units

\[
\nabla^2 \Phi = -\beta \sum_i z_i \alpha_i \exp(-z_i \Phi), \tag{9}
\]

where

\[
\beta = q \frac{L^2 F N_A}{\epsilon RT} C_0, \tag{10}
\]

and \( C_0 \) is a characteristic concentration, \( L \) is the length of a one dimensional domain. What is \( \alpha_i \)? For parameter values, use that \( \epsilon = 6.938 \times 10^{-10} \text{C}^2/\text{Nm}^2 \) at 25C, \( q = 1.6 \times 10^{-19} \text{Coulombs} \), \( \frac{RT}{F} = 25.8 \text{mV} \). How large is \( \beta \) for this cell?

Use that the net charge

\[
\delta = \sum_i z_i \alpha_i. \tag{11}
\]

is a small dimensionless parameter to find the asymptotic solution of the problem.

4. Approximate the time it takes for a projectile whose terminal velocity is \( V_T \) with height \( y(t) \) above the earth, with \( y(0) = 0 \) and \( y_t(0) = V_0 \), to return to earth, assuming that \( \frac{V_0^2}{gR} \) and \( \frac{V_0}{V_T} \) are small. What is the effect of drag on the return time?

Hint: Use the equation

\[
y_{tt} = -\frac{g}{(1 + \frac{y}{R})^2} - \frac{gR}{V_T(R+y)} y_t. \tag{12}
\]

to account for viscous atmospheric drag.

1.1 Solutions

1. (a) There are three solutions,

\[
x = \frac{1}{3} + \frac{1}{81} \epsilon + \frac{1}{729} \epsilon^2 + O(\epsilon^3), \tag{13}
\]

and

\[
x = \pm \sqrt{\frac{3}{\epsilon}} - \frac{1}{6} + \frac{\sqrt{3}}{72} \sqrt{\epsilon} + O(\epsilon). \tag{14}
\]

(b) Take the logarithm of both sides of the equation to rewrite the equation as

\[
(3 + \epsilon) \ln x + \ln(1 + \frac{2\epsilon}{x}) + 2k\pi i. \tag{15}
\]
A power series solution in $\epsilon$ of the form $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$ yields $x_0 = \exp(\pm i \pi / 3)$ or $x = 1$ and $x_1 = -2^3 - x_0^3 \ln(x_0)$, and $x_2 = \frac{1}{6x_0} (3x_1^2 + 4x_1 + 4) - \frac{x_1}{3}$.

For $k = 0$, this gives
\[ x = 1 - \frac{2}{3} \epsilon + \frac{2}{3} \epsilon^2 + O(\epsilon^3). \] (16)

For $k = \pm 1$, the answer is tedious to write out.

(c) To find an asymptotic expansion for $x \exp(-x) = \epsilon$,
\[ y \approx -\eta + \gamma \ln(1 - \eta) + \frac{\ln(1 - \eta)}{(1 - \eta)^2} \gamma^2 + \left( \frac{\ln(1 - \eta)}{(1 - \eta)^2} - \frac{1}{2} \left( \frac{(\ln(1 - \eta))^2}{(1 - \eta)^2} - \frac{1}{2} \frac{(\ln(1 - \eta))^2}{(1 - \eta)^2} \right) \right) \gamma^3 + O \left( \frac{(\ln(1 - \eta))^3}{1 - \eta} \right) \gamma^4 \] (23)

This is an asymptotic series, but a strange one.

(d) There are three solutions
\[ x = 1 + \epsilon - 2 \epsilon^2 + O(\epsilon^3), \] (24)
and
\[ x = \pm i \sqrt{\epsilon} - \frac{1}{2} \epsilon \mp \frac{5}{8} i \sqrt{\epsilon^3}. \] (25)

2. (a) $y(x) = \sin(x) + \epsilon \left( \frac{1}{4} \sin(x) - \frac{1}{8} \sin(x) \cos^2(x) + \frac{3}{8} x \cos(x) + \frac{2}{8} \sin(x) \right)$. (26)

(b) $y(x) = \frac{\sinh(t)}{\sinh(1)} + \epsilon \left( \frac{t \cosh(t)}{\sinh(1)} - \frac{\cosh(1) \sinh(t)}{\sinh^2(1)} \right) + O(\epsilon^2)$, (27)
where $t = \frac{x}{1 + \epsilon}$. 

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3. Take \( z_1 = 1 \) and \( z_2 = -1 \) for sodium and chloride, respectively, and then \( \delta = \alpha_2 - \alpha_1 \). It follows that

\[
\nabla^2 \Phi = -\beta((-\alpha_1 \exp(-\Phi) + (\delta + \alpha_1) \exp(\Phi))).
\]

(28)

Set \( \Phi = \delta \Phi + O(\delta^2) \) and find that to leading order in \( \delta \),

\[
\nabla^2 \Phi_1 + 2 \beta \alpha_1 \Phi_1 = -\beta
\]

(29)

The solution of this equation is easy, being

\[
\Phi_1 = -\frac{1}{2\alpha_1} \left(1 - \frac{\cosh(\sqrt{2\beta \alpha_1}(x - \frac{1}{2}))}{\cosh(\frac{1}{2}\sqrt{2\beta \alpha_1})}\right).
\]

(30)

Now suppose that the maximum value of \( \Phi \) is \( \Phi^* \). This implies that

\[
\frac{\delta}{2\alpha_1} = -\Phi^*.
\]

(31)

Now by conservation, it must be that

\[
1 = \int_0^1 \alpha_2 \exp(\Phi)dx \approx \int_0^1 \alpha_2 (1 + \delta \Phi_1)dx,
\]

(32)

and since

\[
\int_0^1 \Phi_1 dx = -\frac{1}{2\alpha_1} \left(1 - 2 \frac{\tanh(\frac{1}{2}\sqrt{2\beta \alpha_1})}{\sqrt{2\beta \alpha_1}}\right).
\]

(33)

\[
1 = \alpha_2 \left(1 - \frac{1}{2\alpha_1} \left(1 - 2 \frac{\tanh(\frac{1}{2}\sqrt{2\beta \alpha_1})}{\sqrt{2\beta \alpha_1}}\right)\right)
\]

(34)

Now we use that \( \beta \) is very large to get

\[
1 = \alpha_2 \left(1 - \frac{1}{2\alpha_1}\right) = \alpha_2 (1 + \Phi^*).
\]

(35)

Similarly, total sodium (in dimensionless units) is

\[
1 + \Delta_{Na} = \alpha_1 \left(1 + \frac{1}{2\alpha_1}\right) = \alpha_1 (1 - \Phi^*)
\]

(36)

We can now use the three equations (31,35,36) to find that

\[
\Delta_{Na} = \frac{\Phi^*}{(1 + \Phi^*)(1 - 2\Phi^*)}.
\]

(37)

Convert to dimensional units: \( \Phi^* = \frac{\alpha_1}{25.8} = 0.00387 \)

\[
\Delta_{Na} = \frac{\Phi^*}{(1 + \Phi^*)(1 - 2\Phi^*)} = 3.0 \times 10^{-5}.
\]

(38)

This is the fractional charge deflection necessary. Thus, if the initial concentration is 400mM, then the charge deflection concentration is 12\( \mu \)M.
4. Begin by non-dimensionalizing the equation

\[ y'' - \frac{g}{(1 + \frac{y}{R})^2} - \frac{gR}{V_T(R + y)} y' \tag{39} \]

with initial data

\[ y(0) = 0, \quad y_t(0) = V_0. \tag{40} \]

Rescale time \( t = \sigma \tau \) and length \( y = LY \) so that

\[ LY'' = -\frac{g\sigma^2}{(1 + \frac{LY}{R})^2} - \frac{\sigma g}{V_T(1 + \frac{LY}{R})} LY', \tag{41} \]

with initial data \( Y_\tau(0) = \frac{V_0\sigma}{L} \). Pick \( L = V_0\sigma \) and \( g\sigma^2 = \frac{L}{g} \), and \( \epsilon = \frac{L}{R} = \frac{V_0^2}{gR} \). The rescaled equation is

\[ Y'' = -\frac{1}{(1 + \epsilon Y)^2} - \frac{V_0}{V_T} \frac{1}{(1 + \epsilon Y)} Y', \tag{42} \]

with initial data \( Y_\tau(0) = 1 \).

Now set \( \frac{V_0}{V_T} = \gamma \epsilon \), expand in \( \epsilon \), and find that

\[ Y(\tau) = -\frac{1}{2} \tau^2 + t + \epsilon(-\frac{1}{12} \tau^4 + (\frac{1}{6}(\gamma + 2))\tau^3 - \frac{1}{2} \gamma \tau^2) \tag{43} \]

and the root is

\[ \tau = 2 + \epsilon\left(\frac{4}{3} - \gamma \frac{2}{3}\right). \tag{44} \]

Convert this back into dimensional parameters to find

\[ t = 2\frac{V_0}{g} + \frac{4}{3} \frac{V_0^3}{g^2 R} - \frac{2}{3} \frac{V_0^2}{gV_T}. \tag{45} \]

Notice that the correction due to drag decreases the time of flight.

2 Matched Asymptotic Expansions

(due Oct. 8, 2015)

1. Find a composite expansion for the solutions of each of the following (include a plot of the approximate solution);

(a) \( \epsilon y'' + y' = a(>0), \quad y(0) = 0, \quad y(1) = 1, \tag{46} \)

(b) \( \epsilon y'' + 2y' + y^3 = 0, \quad y(0) = 0, \quad y(1) = \frac{1}{2} \tag{47} \)
\[ \epsilon y'' - y' - y^2 = 1, \quad y(0) = \frac{1}{3}, \quad y(1) = 1 \]  
(48)

\[ \epsilon y'' = yy' - y^3, \quad y(0) = \frac{3}{5}, \quad y(1) = -\frac{2}{3} \]  
(49)

\[ \epsilon y'' = 9 - (y')^2, \quad y(0) = 0, \quad y(1) = 1 \]  
(50)

2.1 Solutions

1. (a)
\[ y(x) = ax + (1 - a)(1 - e^{-\frac{2x}{\epsilon}}) + O(\epsilon). \]  
(51)

(b)
\[ y(x) = \frac{1}{\sqrt{3} + x} - \frac{1}{\sqrt{3}} e^{-\frac{2x}{\epsilon}} + O(\epsilon). \]  
(52)

(c)
\[ y(x) = \tan(\tan^{-1}\left(\frac{1}{3}\right) - x) + (1 - \tan(\tan^{-1}\left(\frac{1}{3}\right) - 1)) e^{\frac{2x}{\epsilon}} \]  
(53)

(d) There are three solutions
   i. 
   \[ y_1(x) = \frac{1}{\frac{5}{3} - x} - \frac{3}{2} \tanh\left(\frac{3}{2} \frac{x - 1}{\epsilon} + \tan^{-1}\frac{4}{9}\right) - \frac{3}{2}. \]  
   (54)

   ii. 
   \[ y_2(x) = \frac{4x}{1 + 2x} - 2 \tanh\left(\frac{2x}{\epsilon} - \tan^{-1}\left(\frac{3}{10}\right)\right) \]  
   (55)

   iii. 
   \[ y_3(x) = H(x - \frac{7}{12})\left(\frac{12}{13} \frac{1}{2 + x}\right) + H\left(\frac{7}{12} - x\right)\left(\frac{1}{\frac{5}{3} - x} - \frac{12}{13}\right) - \frac{12}{13} \tanh\left(\frac{12}{13} \frac{x - \frac{7}{12}}{\epsilon}\right) \]  
   (56)

(e)
\[ y(x) = -1 + \epsilon^{\frac{3}{2}} \ln\left(\cosh\left(\frac{3x - 1}{\epsilon^2}\right)\right) \]  
(57)

3 QSS Analysis

(due Nov. 12, 2015)

1. The calcium-activated potassium channel has two open states and two closed states (see Fig. 1). The channel has two binding sites for calcium and can open when one of the states is occupied and remains open when the second site is bound. The binding process is a fast process, while the transition between open and closed states is slow. Find the equation describing slow dynamics of the opening of the channel as a function of calcium concentration.
2. Suppose that an enzyme can bind two substrate molecules, so can exist in one of three states, a free molecule $S$, a complex with one molecule bound $C_1$, and a complex with two molecules bound $C_2$. The corresponding chemical reactions are

$$S + E \xrightarrow{k_1} C_1 \xrightarrow{k_2} P + E$$  \hspace{1cm} (58)$$

$$S + C_1 \xrightarrow{k_3} C_2 \xrightarrow{k_4} P + C_1$$  \hspace{1cm} (59)$$

Give a “proper” derivation of the corresponding Michaelis-Menten rate of production of product $P$, using the assumption that the amount of substrate is much larger than the amount of complex.

3. Consider the process of autocatalytic conversion of a substrate to an enzyme given by

$$S + E \xrightarrow{k_1} C \xrightarrow{k_2} E + E$$  \hspace{1cm} (60)$$

Find the (slow) rate of substrate conversion under the assumption that $k_2 \ll k_{-1}$.

4. Suppose a particle randomly switches between three states

$$S_{-1} \xrightarrow{k} S_0 \xrightarrow{k} S_1,$$  \hspace{1cm} (61)$$

and in states $S_j$ it moves with velocity $jV$, $j = -1, 1$, while in state $S_0$ the particle diffuses with diffusion coefficient $D$. Suppose the switching rates $k$ are quite large. Find the effective diffusion coefficient for this particle.

### 3.1 Solutions

1. The differential equation system is

$$\frac{dp_{c1}}{dt} = -k_1 p_{c1} + k_{-1} p_{c2},$$  \hspace{1cm} (62)$$

$$\frac{dp_{c2}}{dt} = k_1 p_{c1} - k_{-1} p_{c2} - k_2 p_{c2} + k_{-2} p_{o1},$$  \hspace{1cm} (63)$$

$$\frac{dp_{o1}}{dt} = k_2 p_{c2} - k_{-2} p_{o1} - k_3 p_{o1} + k_{-3} p_{o2},$$  \hspace{1cm} (64)$$

$$\frac{dp_{o2}}{dt} = k_3 p_{o1} - k_{-3} p_{o2},$$  \hspace{1cm} (65)$$
where \( k_1 \) and \( k_3 \) are calcium dependent, \( k_1, k_{-1}, k_3, \) and \( k_{-3} \) are large compared to \( k_2, \) and \( k_{-2}. \) Set \( y = p_{c_1} + p_{c_2}. \) The qss approximation implies

\[
0 = -k_1 p_{c_1} + k_{-1} p_{c_2}, \quad (66)
\]

\[
0 = k_3 p_{o_1} - k_{-3} p_{o_2}, \quad (67)
\]

which implies that

\[
p_{c_1} = \frac{k_{-1} y}{k_1 + k_{-1}}, \quad p_{c_2} = \frac{k_1 y}{k_1 + k_{-1}},
\]

\[
p_{o_1} = \frac{k_{-3}}{k_3 + k_{-3}}(1 - y), \quad p_{o_2} = \frac{k_3}{k_3 + k_{-3}}(1 - y).
\]

This implies that

\[
\frac{dy}{dt} = -k_2 p_{c_2} + k_{-2} p_{o_1},
\]

\[
= -\frac{k_2 k_1}{k_1 + k_{-1}} y + \frac{k_{-2} k_{-3}}{k_3 + k_{-3}} (1 - y),
\]

which is the slow equation for the closed-open transition.

2. The differential equation system is

\[
\frac{ds}{dt} = -k_1 e s + k_{-1} c_1 - k_3 s c_1 + k_{-3} c_2, \quad (72)
\]

\[
\frac{de}{dt} = -k_1 e s + k_{-1} c_1 + k_2 c, \quad (73)
\]

\[
\frac{dc_1}{dt} = k_1 e s - k_{-1} c_1 - k_2 c_1 - k_3 s c_1 + k_{-3} c_2 + k_4 c_2, \quad (74)
\]

\[
\frac{dc_2}{dt} = k_3 s c_1 - k_{-3} c_2 - k_4 c_2, \quad (75)
\]

\[
\frac{dp}{dt} = k_2 c_1 + k_4 c_2. \quad (76)
\]

Conservation implies that \( e + c_1 + c_2 = E_0 \) so that we eliminate \( e \) and find

\[
\frac{ds}{dt} = -k_1 s (E_0 - c_1 - c_2) + k_{-1} c_1 - k_3 s c_1 + k_{-3} c_2, \quad (77)
\]

\[
\frac{dc_1}{dt} = k_1 s (E_0 - c_1 - c_2) - k_{-1} c_1 - k_2 c_1 - k_3 s c_1 + k_{-3} c_2 + k_4 c_2, \quad (78)
\]

\[
\frac{dc_2}{dt} = k_3 s c_1 - k_{-3} c_2 - k_4 c_2. \quad (79)
\]

Scale the variables using \( c_1 = E_0 u_1, \ c_2 = E_0 u_2, \ s = S_0 \sigma \) and \( t = \frac{s_0}{k_{-1} E_0} \tau \) and find

\[
\frac{d\sigma}{d\tau} = -\kappa_1 \sigma (1 - u_1 - u_2) + u_1 - \kappa_3 \sigma u_1 + \kappa_{-3} u_2, \quad (80)
\]

\[
\varepsilon \frac{d u_1}{d\tau} = \kappa_1 \sigma (1 - u_1 - u_2) - u_1 - \kappa_2 u_1 - \kappa_3 \sigma u_1 + \kappa_{-3} u_2 + \kappa_4 u_2, \quad (81)
\]

\[
\varepsilon \frac{d u_2}{d\tau} = \kappa_3 \sigma u_1 - \kappa_{-3} u_2 - \kappa_4 u_2, \quad (82)
\]
where \( \kappa_1 = \frac{S_0 k_1}{k_{-1}} \), \( \kappa_2 = \frac{k_2}{k_{-1}} \), \( \kappa_3 = \frac{S_0 k_3}{k_{-1}} \), \( \kappa_{-3} = \frac{k_{-3}}{k_{-1}} \), \( \kappa_4 = \frac{k_4}{k_{-1}} \), and \( \epsilon = \frac{E_0}{S_0} \). The quasi-steady approximation takes

\[
0 = \kappa_1 \sigma (1 - u_1 - u_2) - u_1 - \kappa_2 u_1 - \kappa_3 \sigma u_1 + \kappa_{-3} u_2 + \kappa_4 u_2, \tag{83}
\]

\[
0 = \kappa_3 \sigma u_1 - \kappa_{-3} u_2 - \kappa_4 u_2 \tag{84}
\]

which we solve to find

\[
u_1 = \frac{(\kappa_{-3} + \kappa_4) \kappa_1 \sigma}{\kappa_1 \kappa_3 \sigma^2 + \kappa_1 \sigma (\kappa_{-3} + \kappa_4) + (\kappa_2 + 1)(\kappa_4 + \kappa_{-3})}, \tag{85}\]

\[
u_2 = \frac{\kappa_1 \kappa_3 \sigma^2}{\kappa_1 \kappa_3 \sigma^2 + \kappa_1 \sigma (\kappa_{-3} + \kappa_4) + (\kappa_2 + 1)(\kappa_4 + \kappa_{-3})}. \tag{86}\]

It follows that

\[
\frac{dp}{dt} = E_0 \kappa_1 \frac{k_2 (\kappa_{-3} + \kappa_4) \sigma + k_4 \kappa_3 \sigma^2}{\kappa_1 \kappa_3 \sigma^2 + \kappa_1 \sigma (\kappa_{-3} + \kappa_4) + (\kappa_2 + 1)(\kappa_4 + \kappa_{-3})}. \tag{87}\]

3. The differential equation system is

\[
\frac{ds}{dt} = -k_1 es + k_{-1} c, \tag{88}\]

\[
\frac{de}{dt} = -k_1 es + k_{-1} c + k_2 c, \tag{89}\]

\[
\frac{dc}{dt} = k_1 es - k_{-1} c - k_2 c. \tag{90}\]

Set \( e = E_0 - c \), so that

\[
\frac{ds}{dt} = -k_1 s (E_0 - c) + k_{-1} c, \tag{91}\]

\[
\frac{dc}{dt} = k_1 s (E_0 - c) - k_{-1} c - k_2 c. \tag{92}\]

Under the assumption that \( k_2 \ll k_{-1} \), we take \( s \) to be in quasi-equilibrium so that

\[
\frac{k_1 E_0 s}{k_1 s + k_{-1}}, \tag{93}\]

and then the slow evolution is

\[
\frac{d(s + c)}{dt} = -k_2 c, \tag{94}\]

or

\[
\frac{d}{dt} \left( s + E_0 \frac{s}{s + K_d} \right) = -k_2 \frac{E_0 s}{s + K_d}, \tag{95}\]

where \( K_d = \frac{k_{-1}}{k_1} \), which is a differential equation for the (slow) conversion of substrate to enzyme.
4. The deterministic part of the motion is
\[ \frac{dy}{dt} = Vn \] (96)
where \( n \) has three states \( n = -1, 0, 1 \) with transitions
\[ S_{-1} \xrightarrow{k} S_0 \xrightarrow{k} S_1 \] (97)

The master equations are
\[ \frac{\partial p_{-1}}{\partial t} = kp_0 - kp_{-1} + V \frac{\partial p_{-1}}{\partial x}, \] (98)
\[ \frac{\partial p_0}{\partial t} = kp_{-1} + kp_1 - 2kp_0 + D \frac{\partial^2 p_0}{\partial x^2}, \] (99)
\[ \frac{\partial p_1}{\partial t} = kp_0 - kp_1 - V \frac{\partial p_1}{\partial x}. \] (100)

We assume that \( k \) is large so introduce the parameter \( \epsilon = \frac{1}{k} \). Then, the equations become
\[ \frac{\partial p_{-1}}{\partial t} = \frac{1}{\epsilon} (p_0 - p_{-1}) + V \frac{\partial p_{-1}}{\partial x}, \] (101)
\[ \frac{\partial p_0}{\partial t} = \frac{1}{\epsilon} (p_{-1} + p_1 - 2p_0) + D \frac{\partial^2 p_0}{\partial x^2}, \] (102)
\[ \frac{\partial p_1}{\partial t} = \frac{1}{\epsilon} (p_0 - p_1) - V \frac{\partial p_1}{\partial x}. \] (103)

The nullspace of the matrix is spanned by the vector
\[ \phi = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \] (104)

We set
\[ p = \begin{pmatrix} p_{-1} \\ p_0 \\ p_1 \end{pmatrix} = v\phi + \begin{pmatrix} w_{-1} \\ w_0 \\ w_1 \end{pmatrix}, \] (105)
and find the equation for \( v \) by projecting
\[ \frac{\partial v}{\partial t} = V \frac{\partial p_{-1}}{\partial x} - V \frac{\partial p_1}{\partial x} + D \frac{\partial^2 p_0}{\partial x^2} \] (106)
\[ = V \frac{\partial}{\partial x} (w_{-1} - w_1) + \frac{1}{3} D \frac{\partial^2}{\partial x^2} (v + w_0). \] (107)

The vector \( w \) must satisfy the equation
\[ \frac{1}{\epsilon} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} w_{-1} \\ w_0 \\ w_1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -V \frac{\partial v}{\partial x} \\ 0 \\ V \frac{\partial v}{\partial x} \end{pmatrix}. \] (108)
The solution of this problem has

$$w_{-1} - w_1 = \frac{2\epsilon}{3} V \frac{\partial v}{\partial x} .$$  \hfill (109)$$

Thus, the effective Fokker-Planck equation (ignoring higher order derivatives) is

$$\frac{\partial v}{\partial t} = D_{eff} \frac{\partial^2 v}{\partial x^2}$$  \hfill (110)$$

where

$$D_{eff} = \frac{1}{3} D + \frac{2\epsilon}{3} V^2 ,$$  \hfill (111)$$

which, in case you are checking, has the correct units.