Math 1220-3  Mock Exam 2

Name: _______________________________________

Show all work. Write your answer in the space provided.

1. (4 pts) Find \( \lim_{x \to 0} \frac{4x}{\tan x} \).

**Solution**

This is an indeterminate form of type 0/0. So we need to use L'Hopital's rule to find the limit. Then

\[
\lim_{x \to 0} \frac{4x}{\tan x} = \lim_{x \to 0} \frac{4}{\sec^2 x} = \lim_{x \to 0} 4 \cos^2 x = 4.
\]

2. (5pts) Find \( \lim_{x \to 0} x^x \).

**Solution**

This is an indeterminate form of the type 0^0. So we need to transform this into either an indeterminate of the form \( \infty/\infty \) or an indeterminate form of the type 0/0. We do this by letting \( y = x^x \) and then taking logs of both sides to obtain \( \ln y = x \ln x \). Now multiply both sides by -1 to obtain \(-\ln y = -x \ln x\). Now let us find the limit of \(-x \ln x\) as \( x \) goes to 0. We can rewrite this as

\[
-\frac{\ln x}{x}.
\]

The limit as \( x \) tends to 0 of this expression is a indeterminate form of type \( \infty/\infty \), so we can use L'Hopital's rule to find the limit. L'Hopital's rule tells us the limit should be the same as the limit of

\[
-x^{-1} - x^{-2} = x,
\]

as \( x \) tends to 0. Hence as \( x \) tends to 0, \(-\ln y \) will tend to 0. Now we use the fact that \( y = e^{\ln y} \). So as \( x \) converges to 0, \( y \) will tend to \( e^0 = 1 \).

3. (5pts) Show that the improper integral \( \int_0^\infty \frac{dx}{1 + x^2} \) converges.

**Solution**

To do this we need to show that \( \lim_{b \to \infty} \int_1^b \frac{dx}{1 + x^2} \) converges. Now

\[
\lim_{b \to \infty} \int_1^b \frac{dx}{1 + x^2} = \lim_{b \to \infty} \left[ \arctan x \right]_0^b = \lim_{b \to \infty} \arctan b = \frac{\pi}{2}.
\]
Hence this shows that \( \lim_{b \to \infty} \int_1^b \frac{dx}{1 + x^2} \) converges.

4. (6pts) Find the area of the region under the curve \( y = \frac{1}{x^2 + x} \) to the right of \( x = 1 \).

\[ \text{Solution} \]

We need to find the integral \( \int_1^\infty \frac{dx}{x^2 + x} \). The first thing we must do is use partial fractions to rewrite \( \frac{1}{x^2 + x} \). Then \( \frac{1}{x^2 + x} = \frac{1}{x} - \frac{1}{x + 1} \). Hence,

\[
\int_1^\infty \frac{1}{x^2 + x} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^2 + x} = \lim_{b \to \infty} \int_1^b \left[ \frac{1}{x} - \frac{1}{x + 1} \right] dx
\]

\[
= \lim_{b \to \infty} \left[ \ln x - \ln(x + 1) \right]_1^b
\]

\[
= \lim_{b \to \infty} \left[ \ln(x/(x + 1)) \right]_1^b
\]

\[
= \lim_{b \to \infty} \left[ \ln(b/(b + 1)) - \ln 1/2 \right]
\]

\[
= \ln 2.
\]

5. (6 pts) Find \( \lim_{n \to \infty} a_n \), where \( a_n = \frac{4n^2 + 2}{n^2 + 3n - 1} \).

\[ \text{Solution} \]

\[
\lim_{n \to \infty} \frac{4n^2 + 2}{n^2 + 3n - 1} = \lim_{n \to \infty} \frac{4 + \frac{2}{n^2}}{1 + \frac{3}{n} - \frac{1}{n^2}} = \lim_{n \to \infty} \frac{4 + \frac{2}{n^2}}{\lim_{n \to \infty} 1 + \frac{3}{n} - \frac{1}{n^2}} = \frac{\lim_{n \to \infty} 4 + \lim_{n \to \infty} \frac{2}{n^2}}{\lim_{n \to \infty} 1 + \lim_{n \to \infty} \frac{3}{n} + \lim_{n \to \infty} \frac{1}{n^2}} = 4.
\]

6. (5 pts) Determine whether the series, \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \), converges or diverges. Give a reason for your answer.

\[ \text{Solution} \]
Notice that for large \( n \) the series is behaving like the harmonic series. So we use the limit comparison test. Let \( a_n = \frac{n}{n+1} \) and \( b_n = \frac{1}{n} \). We need to find the limit of \( a_n/b_n \) as \( n \) tends to infinity. Now \( \frac{a_n}{b_n} = \frac{n^2}{n+1} \). The limit of this as \( n \) tends to infinity is 1. We know that the Harmonic series diverges, hence using the limit comparison test, the series \( \sum a_n \) diverges.

7. (6pts) By differentiating the geometric series

\[
\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \ldots, \quad |x| < 1 ,
\]

find a power series that represents \( 1/(1 + x)^2 \). What is the interval of convergence?

Solution

Notice if we take the derivative of \( \frac{1}{1+x} \), we obtain \(-1/(1 + x)^2\). If we differentiate the power series term by term we find that

\[
\frac{-1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 - \ldots, \quad |x| < 1 ,
\]

Now we multiply both sides of the equation by -1 we obtain

\[
\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 - \ldots, \quad |x| < 1 ,
\]

8. (8 pts) Find the first four terms of the Maclaurin series for \( e^x \sin x \)

Solution

Let \( f(x) = e^x \sin x \). We first need to find the first three derivatives of this function

\[
\begin{align*}
 f'(x) &= e^x \sin x + e^x \cos x, \\
 f''(x) &= e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x = 2e^x \cos x, \\
 f'''(x) &= 2(e^x \cos x - e^x \sin x)
\end{align*}
\]

These imply that \( f'(0) = 1, f''(0) = 2 \) and \( f'''(0) = 2 \).

Hence the first three terms of the Maclaurin series is

\[
0 + x + x^2 + \frac{x^3}{3} \ldots
\]

9. (4 pts) Is the following series \( \sum_{n=1}^{\infty} \frac{\cos n \pi}{n} \) conditionally convergent? Explain your answer.

Solution

This is an alternating series. So we need to check if the sequence \( \{ \frac{1}{n} \} \) is a non-negative, decreasing sequence which also converges to 0. Well it clearly satisfies all those properties so the series \( \sum_{n=1}^{\infty} \frac{\cos n \pi}{n} \)
certainly converges. Now we have to check to see if the series \( \sum_{n=1}^{\infty} \frac{\cos n\pi}{n} \) converges. This series is just the harmonic series, and we know that the harmonic series diverges. Hence this series is conditionally convergent.

10. (12 pts) let

\[
f(x) = \begin{cases} 
e^{-1/x^2} & x \neq 0, \\ 0 & x = 0. \end{cases}
\]

(a) Show that \( f'(0) = 0 \) by using the definition of the derivative.

(b) Show that \( f''(0) = 0 \).

(c) Assuming the known fact that \( f^{(n)}(0) = 0 \) for all \( n \), find the Maclaurin series for \( f(x) \).

(d) Does the Maclaurin series represent \( f(x) \).

**Solution**

(a) We need to find the derivative of the function \( f \) at the origin. This by definition is

\[
f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{-1/h^2}}{h}
\]

We can rewrite this as

\[
\lim_{h \to 0} \frac{1/h}{e^{1/h^2}}
\]

Now L’Hopital’s rule tells me that this limit is

\[
\lim_{h \to 0} \frac{h}{2e^{1/h^2}}
\]

which is zero.

(b) We need to find the second derivative of the function \( f \) at the origin. This is by definition

\[
f''(0) = \lim_{h \to 0} \frac{f''(h) - f''(0)}{h} = \lim_{h \to 0} \frac{2h^{-3}e^{-1/h^2}}{h}
\]

We can rewrite this as

\[
\lim_{h \to 0} \frac{2/h^4}{e^{1/h^2}}
\]

You can check, using L’Hopital’s rule, that this limit is 0.

(c) The Maclaurin series for \( f(x) \) is given by \( f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots \frac{f^{(n)}(0)}{n!}x^n + \ldots \) Since \( f^{(n)}(0) = 0 \) for all \( n \). This shows us that the Maclaurin series is just 0.

(d) The Maclaurin series does not represent \( f(x) \). If we choose any \( x \) around the origin, then \( f(x) \) will be non-zero, which is not equal to the Maclaurin series.
11. (6pts) Show that \( \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{1 + (k/n)^2} \cdot \frac{1}{n} = \frac{\pi}{4} \). Hint: write an equivalent definite integral.

Solution

The series is just the integral \( \int_{0}^{1} \frac{dx}{1 + x^2} \). The value of this integral is \([\arctan x]_{0}^{1}\), which is \(\frac{\pi}{4}\).

12. (5pts) Find the convergence set of the power series \( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots \).

Solution

The power series written in sum notation is \( \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!} \). We have to use the absolute ratio test to determine the convergence set. So we need to find the limit of \( \left| \frac{(-1)^{n+2}x^{2n+1}/(2n+1)!}{(-1)^{n+1}x^{2n-1}/(2n-1)!} \right| \). This expression is equal to \( \frac{x^2}{(2n+1)(2n)} \), which converges to 0 as \( n \) converges to infinity. Hence the convergence set is the whole real line.

13. (7pts) Suppose that \( \sum_{n=0}^{\infty} a_n(x - 3)^n \) converges at \( x = -1 \). Why can we conclude that it converges at \( x = 6 \)? Can we be sure that it converges at \( x = 7 \)? Explain your answer.

Solution

Since the series converges at \( x = -1 \), we know that the radius of convergence is at least 4 so it must converge at \( x = 6 \). The radius of convergence could only be 4. In which case we do not know that it converges at the end point of the interval.

14. (7pts) By writing \( \frac{1}{x} = \frac{1}{1-(-x)} \) and using the known expansion of \( \frac{1}{1-x} \), find the Taylor series for \( \frac{1}{x} \) in powers of \( x - 1 \). You only need to write down the first five terms.

Solution

The power series of \( \frac{1}{1-x} \) is given by

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots , \quad |x| < 1 ,
\]

Using this we deduce that the power series of \( \frac{1}{1-(1-x)} \) is

\[
\frac{1}{1-(1-x)} = 1 + (1-x) + (1-x)^2 + (1-x)^3 + (1-x)^4 + \ldots , \quad |x-1| < 1 ,
\]

Hence the Taylor series expansion of \( \frac{1}{x} \) about the point \( x = 1 \) is

\[
\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 + \ldots , \quad |x-1| < 1 ,
\]
15. (7pts) Does the function \( f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \) satisfy the differential equation \( y' + y = 0 \) on the whole real line? Explain your answer.

Solution

Let \( f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \). This series converges on the whole real line (We checked this in class).

Now we take the derivative of \( f(x) \). So \( f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{n-1}}{(n-1)!} \) Now

\[
f'(x) + f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{n-1}}{(n-1)!} = 0.
\]

Since the series converges for every real number this equation is valid for every real number.