
October 25, 2000

1 (10 points). Problem #1, page 50 from the textbook:
Prove that the sum of two Cauchy sequences is again a Cauchy sequence.

Solution. Suppose \((x_n), (y_n)\) are Cauchy sequences. Their sum is \((x_n + y_n) = (z_n)\).

I first give the argument going “backwards”. We want to show that \((z_n)\) is Cauchy. This means that for each \(\epsilon > 0\) we will find \(n_3\) so that for all \(n, m \geq n_3\) we have: \(|z_n - z_m| = |(x_n + y_n) - (x_m + y_m)| < \epsilon\). Note that

\[
|z_n - z_m| = |(x_n - x_m) + (y_n - y_m)| \leq |x_n - x_m| + |y_n - y_m|
\]

(by the triangle inequality). However, \((x_n), (y_n)\) are Cauchy, therefore for each number \(\epsilon/2 > 0\) there exists \(n_1, n_2\) so that:

(i) For all \(n, m \geq n_1\) we have \(|x_n - x_m| < \epsilon/2\).
(ii) For all \(n, m \geq n_2\) \(|y_n - y_m| < \epsilon/2\).

Thus \(|x_n + y_n) - (x_m + y_m)| < \epsilon/2 + \epsilon/2 = \epsilon\) provided that \(n, m \geq n_3 = \max(n_1, n_2)\). Hence \((z_n)\) is Cauchy.

Now, on the basis of the above computations I will give the direct argument. Given \(\epsilon > 0\) take \(\delta = \epsilon/2\). Since \((x_n), (y_n)\) are Cauchy, there exist \(n_1 \in \mathbb{N}, n_2 \in \mathbb{N}\) so that:

(1) \(|x_n - x_m| < \delta\) for all \(n, m \geq n_1\).
(2) \(|y_n - y_m| < \delta\) for all \(n, m \geq n_2\).

Then for all \(n, m \geq n_3 = \max(n_1, n_2)\) we have:

\[|x_n - x_m| < \delta, \quad |y_n - y_m| < \delta.\]

Thus (by the triangle inequality)

\[|(x_n+y_n)-(x_m+y_m)| = |(x_n-x_m)+(y_n-y_m)| \leq |x_n-x_m|+|y_n-y_m| < \delta+\delta = \epsilon.\]

Therefore for each \(\epsilon > 0\) we found \(n_3\) so that for all \(n, m \geq n_3\) we have

\[|z_n - z_m| = |(x_n + y_n) - (x_m + y_m)| < \epsilon.\]

Thus \((z_n)\) is Cauchy. \qed

2 (10 points). Problem #3, page 50:
Suppose \((x_n) \in \mathbb{N}\) for all \(n\). If \((x_n)\) is Cauchy, prove that there are numbers \(a, n_0 \in \mathbb{N}\) so that \(x_n = a\) for all \(n \geq n_0\).

**Solution.** Since \((x_n)\) is Cauchy, for each \(\epsilon > 0\) there exists \(n_0 \in \mathbb{N}\) so that \(n, m \geq n_0 \Rightarrow |x_n - x_m| < \epsilon\). Recall that if \(A, B\) are integers, then \(|A - B| \geq 1\) unless \(A = B\). Thus if we take \(\epsilon = 0.5\) (which is less than 1), then (for \(n_0\) given by the definition of Cauchy sequence) for all \(n, m \geq n_0\) we have:

\[
|x_n - x_m| < \epsilon = 0.5 < 1.
\]

Hence \(x_n = x_m\) is the same natural number (which we denote \(a\)) for all \(n, m \geq n_0\). Thus \(x_n = a\) for all \(n \geq n_0\). \(\square\)

3 (10 points). Problem # 1a, page 62:
Using definition of the limit show that

\[
\lim_{x \to 2} x^2 - x + 1 = 3.
\]

**Solution.** Given \(\epsilon > 0\) we will find \(\delta > 0\) so that for all \(x\) satisfying \(0 < |x - 2| < \delta\) we have \(|x^2 - x + 1 - 3| < \epsilon\). Note that

\[
x^2 - x + 1 - 3 = x^2 - x - 2 = (x - 2)^2 - (x - 2) - 4 = (x - 2)^2 + 4(x - 2) - (x - 2) = (x - 2)^2 + 3(x - 2).
\]

We want the inequality \(|(x - 2)^2 + 3(x - 2)| < \epsilon\). I give the “backwards” argument for that and leave it for you to write the “direct” argument. By the triangle inequality, \(|(x - 2)^2 + 3(x - 2)| \leq |x - 2| + 3|x - 2|\).

Thus, if \(|x - 2|^2 < \epsilon/2\) and \(3|x - 2| < \epsilon/2\) then \(|(x - 2)^2 + 3(x - 2)| < \epsilon\) and we are happy. Now we “solve” each of the two inequalities:

\[
|x - 2|^2 < \epsilon/2 \iff |x - 2| < \sqrt{\epsilon/2},
\]

\[
3|x - 2| < \epsilon/2 \iff |x - 2| < \epsilon/6.
\]

Therefore, if we take \(\delta = \min(\sqrt{\epsilon/2}, \epsilon/6)\) then

\[
0 < |x - 2| < \delta \Rightarrow |x - 2|^2 < \epsilon/2 \text{ and } 3|x - 2| < \epsilon/2.
\]

Thus,

\[
0 < |x - 2| < \delta \Rightarrow |(x - 2)^2 + 3(x - 2)| < \epsilon \Rightarrow |x^2 - x + 1 - 3| < \epsilon.
\]

Alternative solution. Let \(\epsilon > 0\). We will find \(\delta > 0\) so that \(0 < |x - 2| < \delta \Rightarrow |x^2 - x + 1 - 3| < \epsilon\). Factoring the polynomial \(x^2 - x + 1 - 3\) we get:

\[
x^2 - x + 1 - 3 = (x + 1)(x - 2).
\]

We want to find an upper bound for the first factor provided that \(x \in [1, 3]\):

\[
1 \leq x \leq 3 \iff 2 \leq x + 1 \leq 4 \Rightarrow |x + 1| \leq 4.
\]
Thus $|x + 1| \leq 2$ for each $x \in [1, 3]$. Choose $\delta = \min(1, \epsilon/4)$. Then $0 < |x - 1| < \delta$ implies that:

(a) $x \in [1, 3]$ and hence $|x + 1| \leq 4$;
(b) $|x - 2| < \epsilon/4$.

Thus $0 < |x - 1| < \delta$ implies that $|(x + 1)(x - 2)| < 4 \cdot \epsilon/4 = \epsilon$. □

4 (5 points). Problem # 1b, page 63:
Using definition of the limit show that
\[
\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.
\]

**Solution.** First note that $\frac{x^2 - 1}{x - 1}$ is defined for all $x \neq 0$, thus the notion of the limit (at zero) makes sense. $\frac{x^2 - 1}{x - 1} = x + 1$ (for $x \neq 1$). Hence, given $\epsilon > 0$ we need to find $\delta > 0$ so that $0 < |x - 1| < \delta$ implies that $|x + 1 - 2| < \epsilon$. The latter inequality is equivalent to $|x - 1| < \epsilon$. Thus we can take $\delta = \epsilon$. Then

\[
0 < |x - 1| < \delta = \epsilon \Rightarrow |x - 1| < \epsilon \Rightarrow \left| \frac{x^2 - 1}{x - 1} - 2 \right| < \epsilon. \quad \square
\]

5 (10 points). Problem # 2a, page 63: Decide if the following limit exists:
\[
\lim_{x \to 0} \cos(1/x).
\]

**Solution.** Let $f(x)$ denote the function $\cos(1/x)$. Note that this function is defined for all $x \neq 0$, hence the notion of the limit at zero makes sense. We will show that the limit $\lim_{x \to 0} f(x)$ does not exist. To prove this we use Theorem 3.6. We will construct two sequences $(x_n)$, $(y_n)$ convergent to zero so that the limits of the sequences $(\cos(1/x_n))$ and $(\cos(1/y_n))$ are different.

Note that $\cos(2\pi n) = 1$ for each $n \in \mathbb{Z}$. Thus, if we take $x_n = \frac{1}{2\pi n}$ (where $n \in \mathbb{N}$) then $f(x_n) = \cos(1/x_n) = \cos(2\pi n) = 1$. Hence the sequence $f(x_n)$ is constant (equal to 1) and therefore converges to 1. On the other hand, $x_n = \frac{1}{2\pi n}$ converges to zero (since this is the quotient of a bounded sequence by a sequence which diverges to infinity). Hence we found the first sequence.

To find the second sequence note that $\cos(2\pi n + \pi/2) = 0$ for each $n \in \mathbb{Z}$. Thus, if we take $y_n = \frac{1}{2\pi n + \pi/2}$ (where $n \in \mathbb{N}$) then $f(y_n) = \cos(1/y_n) = \cos(2\pi n + \pi/2) = 0$. Hence the sequence $f(y_n)$ is constant (equal to zero) and therefore converges to 0. On the other hand, $y_n = \frac{1}{2\pi n + \pi/2}$ converges to zero (since this is the quotient of a bounded sequence by a sequence which diverges to infinity). Thus we found the second sequence.

Therefore, by Theorem 3.6, the limit $\lim_{x \to 0} f(x)$ does not exist. □