MATH 3100. 3-rd Midterm Test: Solutions.

April 17, 2003

1. [20 points] Given points $P = (1, 2)$ and $Q = (2, 3)$ find a motion $m$ of the plane so that $m((0, 0)) = P$ and $m((\sqrt{2}, 0)) = Q$.

Solution. We first observe that $d = d(P, Q) = \sqrt{(2 - 1)^2 + (3 - 2)^2} = \sqrt{2}$. The angle between the line $(PQ)$ and the $x$-axis equals $\alpha = 45^\circ$. The translation by $(-1, -2)$, $T(z) = z - (1 + 2i)$ moves the point $P$ to the origin and moves the point $Q$ to the point $(2 - 1, 3 - 2) = (1, 1)$. Rotation $R_\alpha$ by the angle $-45^\circ$ moves the point $(1, 1)$ to the point $(\sqrt{2}, 0)$. Thus $R_\alpha \circ T(z) = e^{-i\pi/4}(z - (1 + 2i))$ moves $P$ to $(0, 0)$ and $Q$ to $(\sqrt{2}, 0)$.

We now reverse this process: First apply the rotation $R_\alpha$ and then translation $T^{-1}$ by $(1, 2)$: the resulting motion $m$ will send $(0, 0)$ to $P$ and $(\sqrt{2}, 0)$ to $Q$. Algebraically, this composition is written as

\[
m(z) = e^{i\pi/4}z + 1 + 2i = (\cos(\pi/4) + i \sin(\pi/4))z + 1 + 2i = (\sqrt{2}/2 + i\sqrt{2}/2)z + 1 + 2i. \]

2. [20 points] Using axioms of the plane prove that given a straight line $L$ and a point $P$ in the plane, there exists a straight line $L'$ through the point $P$ which is parallel to $L$. (You may use without a proof uniqueness of a straight line through the given pair of distinct points, as well as facts about motions of the plane.)

Solution: See the handout.

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3. [20 points] Using the triangle inequality for absolute values of complex numbers prove the triangle inequality for distances between points in the plane.

Alternatively: Prove the triangle inequality for distances between points in the plane (following the textbook).

Solution: See the handout or the textbook.

4. [20 points] (a) State the definition of a regular $n$-gon (in the plane).
(b) Suppose that $P$ is a regular $n$-gon whose sides have length 1. Compute (in terms of $n$) the radius of the inscribed circle in $P$. Justify your answer!

Solution: (a) A regular $n$-gon is a polygon where all $n$ sides are equal and all $n$ angles are equal.

(b) Let’s compute the angle $\alpha = \angle AOC$ from the center of the polygon (see Figure): the full angle around the center equals $2\pi$, hence $\alpha = 2\pi/n$. Half of this angle equals $\pi/n$. Consider the midpoint $B$ of the side $AC$ of the polygon as on the Figure. Since the triangle $ABC$ is isosceles, we have: $OB$ is the altitude of the triangle $ABC$ and hence $|OB| = r$ is the radius of the inscribed circle. We also have $|AB| = 1/2$, hence

$$|AB|/|OB| = \tan(\alpha), |OB| = |AB| \cot(\alpha) = \frac{\cot(\pi/n)}{2}.$$ 

Thus

$$r = |OB| = \frac{\cot(\pi/n)}{2}.$$

Thus

![Figure 1:](image)

5. [20 points] State the classification theorem for motions of the plane.
Solution: See the textbook.