MATHEMATICS 3210-2. First Midterm Test (Sample): Solutions.

January 25, 2002

1. [15 points] Using the definition of the limit of a sequence prove that the following sequence converges:

$$x_k = \left( \frac{k}{k-1}, \frac{1}{k^2} \right).$$

**Solution.** I claim that the limit of this sequence is $(1, 0)$. Let $\epsilon > 0$, we have to find $K$ such that for all $k > K$ we have:

$$\left\| \left( \frac{k}{k-1} - 1, \frac{1}{k^2} \right) \right\| < \epsilon.$$

Since $\|v\| \leq \sqrt{2}\|v\|_\infty$, it suffices to find $K_1, K_2$ such that for all $k > K = \max(K_1, K_2)$ we have:

$$\left| \frac{k}{k-1} - 1 \right| < \frac{1}{\sqrt{2}}, \frac{1}{k^2} < \frac{1}{\sqrt{2}}.$$

To find $K_1, K_2$ we “solve” each of the above inequalities:

$$\left| \frac{k}{k-1} - 1 \right| < \frac{1}{\sqrt{2}} \iff \frac{1}{k-1} < \frac{1}{\sqrt{2}}$$

(assuming $k > 1$), equivalently, $\frac{\sqrt{2}}{\epsilon} < k - 1 \iff \frac{\sqrt{2}}{\epsilon} - 1 < k$.

Hence we can take $K_1 = \max(2, \frac{\sqrt{2}}{\epsilon} - 1)$.

For the second inequality we have:

$$\frac{1}{k^2} < \frac{1}{\sqrt{2}} \iff \frac{\sqrt{2}}{\epsilon} < k^2 \iff$$

$$\frac{2^{1/4}}{\sqrt{\epsilon}} < k.$$  

Hence we take $K_2 := \frac{2^{1/4}}{\sqrt{\epsilon}}$. Then for each $k \geq \max(K_1, K_2)$ we have both

$$\left| \frac{k}{k-1} - 1 \right| < \frac{1}{\sqrt{2}}, \frac{1}{k^2} < \frac{1}{\sqrt{2}}.$$  

$\square$
2. [15 points] State the Bolzano-Weierstrass theorem for $\mathbb{R}^n$.
See the textbook.

3. [20 points] Compute the following limit or show that it does not exist:
$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^2 + y^2}$$
(you can use limit theorems).

**Solution.** Note that $\lim_{(x,y)\to(0,0)} x^2 = 0$. We will verify that the function $\frac{x^2}{x^2 + y^2}$ is bounded. Note that $x^2 \leq x^2 + y^2$, hence $\frac{x^2}{x^2 + y^2} \leq 1$. Thus
$$0 \leq \frac{x^4}{x^2 + y^2} \leq x^2.$$ 
Since the function $x^2$ is continuous, $\lim_{(x,y)\to(0,0)} x^2 = 0$, thus by the squeeze theorem,
$$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^2 + y^2} = 0.$$

4. [15 points] Let $f(x, y) = (x, y^2)$. Using the definition of total derivative verify that
$$Df(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & 2y \end{bmatrix}.$$ 
**Solution.** Let $a = (a, b) \in \mathbb{R}^2$. Let 
$$T = \begin{bmatrix} 1 & 0 \\ 0 & 2b \end{bmatrix}.$$ 
Then we have to verify:
$$\lim_{h \to 0} \frac{\|f(a + h) - f(a) - Th\|}{\|h\|} = 0.$$ 
The vector in the numerator is:
$$\begin{bmatrix} a + x \\ (b + y)^2 \end{bmatrix} - \begin{bmatrix} a \\ b^2 \end{bmatrix} - \begin{bmatrix} x \\ 2by \end{bmatrix} = \begin{bmatrix} 0 \\ y^2 \end{bmatrix}.$$ 
The norm of this vector is $y^2$. The norm of the vector in the denominator is
$$\|h\| = \sqrt{x^2 + y^2}.$$ 
Hence we have to show that
$$\lim_{x \to 0, y \to 0} \frac{y^2}{\sqrt{x^2 + y^2}} = 0.$$ 
Note that
$$\lim_{x \to 0, y \to 0} \frac{y^4}{x^2 + y^2} = 0.$$
(see problem 3). Hence, since the square root is a continuous function,

\[
\lim_{x \to 0, y \to 0} \frac{y^2}{\sqrt{x^2 + y^2}} = \lim_{x \to 0, y \to 0} \frac{y^4}{x^2 + y^2} = 0. \quad \square
\]

5. [15 points] Determine if the subset \( \{(x, y) : x = 0, y \in \mathbb{R}\} \) of \( \mathbb{R}^2 \) is open. Give a proof!

Solution. I claim that this set \( A \) is not open. Take \( p = (0, 0) \in A \). For each \( \epsilon > 0 \) the open ball \( B_\epsilon(p) \) will contain the point

\[
(\epsilon/2, 0)
\]

which does not belong to \( A \). \( \square \)

6. [20 points] Reorder the following sentences to get a valid proof of the theorem on uniqueness of limit of a function:

Proof. By the triangle inequality we get:

\[
\|v - w\| \leq \|f(x) - v\| + \|f(x) - w\| < 2\epsilon = \|v - w\|.
\]

Suppose \( \lim_{x \to a} f(x) = v, \lim_{x \to a} f(x) = w \). Suppose that \( v \neq w \), then \( \|v - w\| > 0 \). Thus \( \|v - w\| < \|v - w\| \). Contradiction. Hence for all \( x \) satisfying \( \|x - a\| < \min(\delta_1, \delta_2) \) we have:

\[
\|f(x) - v\| < \epsilon, \|f(x) - w\| < \epsilon.
\]

Let \( \epsilon = \|v - w\|/2 \). Then (since \( \lim_{x \to a} f(x) = v \), there exists \( \delta_1 > 0 \) such that \( 0 < \|x - a\| < \delta_1 \) implies \( \|f(x) - v\| < \delta_1 \). Similarly, (since \( \lim_{x \to a} f(x) = w \), there exists \( \delta_2 > 0 \) such that \( 0 < \|x - a\| < \delta_2 \) implies \( \|f(x) - w\| < \delta_2 \). \( \square \)

Solution. Suppose \( \lim_{x \to a} f(x) = v, \lim_{x \to a} f(x) = w \). Suppose that \( v \neq w \), then \( \|v - w\| > 0 \). Let \( \epsilon = \|v - w\|/2 \). Then (since \( \lim_{x \to a} f(x) = v \), there exists \( \delta_1 > 0 \) such that \( 0 < \|x - a\| < \delta_1 \) implies \( \|f(x) - v\| < \delta_1 \). Similarly, (since \( \lim_{x \to a} f(x) = w \), there exists \( \delta_2 > 0 \) such that \( 0 < \|x - a\| < \delta_2 \) implies \( \|f(x) - w\| < \delta_2 \). Hence for all \( x \) satisfying \( \|x - a\| < \min(\delta_1, \delta_2) \) we have:

\[
\|f(x) - v\| < \epsilon, \|f(x) - w\| < \epsilon.
\]

By the triangle inequality we get:

\[
\|v - w\| \leq \|f(x) - v\| + \|f(x) - w\| < 2\epsilon = \|v - w\|.
\]

Thus \( \|v - w\| < \|v - w\| \). Contradiction. \( \square \)