
1. [15 points] Using the definition of the limit of a sequence prove that the following sequence converges:

\[ x_k = \left( \frac{k}{k+1}, \frac{(-1)^k}{k^2} \right). \]

Solution. We will show that the limit of this sequence equals \((1,0)\), in other words, for each \(\varepsilon > 0\) we want to find \(n_0 \in \mathbb{N}\) such that for all \(k \geq n_0 + 1,

\[ \varepsilon > \| (\frac{k}{k+1} - 1, \frac{(-1)^k}{k^2}) \| = \| (\frac{-1}{k+1}, \frac{(-1)^k}{k^2}) \|. \]

Since \(\|x\| \leq \sqrt{2}\|x\|_{\infty}\), it suffices to find \(n_0\) so that for all \(k \geq n_0\),

\[ \frac{1}{k+1} < \varepsilon/\sqrt{2}, \frac{1}{k^2} < \varepsilon/\sqrt{2}. \] (1)

Let’s solve both inequalities for \(k\):

\[ \frac{1}{k+1} < \varepsilon/\sqrt{2} \iff \frac{\sqrt{2}}{\varepsilon} < k + 1, \]

hence for the first inequality it suffices to take \(k \geq n_1 = \lceil \frac{\sqrt{2}}{\varepsilon} \rceil\).

We also have

\[ \frac{1}{k^2} < \varepsilon/\sqrt{2} \iff \frac{\sqrt{2}}{\varepsilon} < k^2 \iff \frac{2^{1/4}}{\sqrt{\varepsilon}} < k. \]

Hence for the second inequality it suffices to take \(k \geq n_2 = \lceil \frac{2^{1/4}}{\sqrt{\varepsilon}} \rceil\).

Hence, by taking \(n_0 = \max(\lceil \frac{2^{1/4}}{\sqrt{\varepsilon}} \rceil, \lceil \frac{\sqrt{2}}{\varepsilon} \rceil)\) we get the required assertion.

2. [15 points] State and prove theorem about equivalence of convergence of a sequence \(x_k\) in \(\mathbb{R}^n\) and convergence of the coordinate sequences \(x_k(j), j = 1,...,n\).

(You can use relation between norm and the sup-norm.)

Solution. See the textbook.
3. [10 points] Compute the iterated limits of

$$f(x, y) = \frac{\sin(x) \sin(y)}{x^2 + y^2}.$$ 

Decide if the function has a limit as \((x, y) \to (0, 0)\) and prove that the limit exists (or does not exist).

Solution. The iterated limits are:

$$\lim_{y \to 0} \lim_{x \to 0} \frac{\sin(x) \sin(y)}{x^2 + y^2} = \lim_{y \to 0} \frac{0}{y^2} = 0,$$

$$\lim_{x \to 0} \lim_{y \to 0} \frac{\sin(x) \sin(y)}{x^2 + y^2} = \lim_{x \to 0} \frac{0}{x^2} = 0.$$

Hence both iterated limits are equal to zero. Now consider the limit along the line \(x = y\):

$$\lim_{x \to 0} \frac{\sin^2(x)}{2x^2} = \frac{1}{2} \lim_{x \to 0} \frac{\sin^2(x)}{x^2}.$$ 

Recall that \(\lim_{x \to 0} \frac{\sin(x)}{x} = 1\) (for instance, from the l’Hospital’s rule). Hence by the product theorem for the limits we get:

$$1 = \lim_{x \to 0} \left(\frac{\sin(x)}{x}\right)^2 = \lim_{x \to 0} \frac{\sin^2(x)}{x^2}.$$ 

Thus

$$\lim_{x \to 0} \frac{\sin^2(x)}{2x^2} = \frac{1}{2} \neq 0.$$ 

Hence the limit along the line \(x = y\) is different from the iterated limits, so the limit of the function does not exist.

4. [10 points] State the Bolzano-Weierstrass theorem for \(\mathbb{R}^n\).

Solution. See the textbook.

5. [15 points] Find the interior of the set

$$E = \{(x, y) \in \mathbb{R}^2 : xy \leq 0\}.$$ 

(You can use theorems about continuous functions here.)

Solution. Let’s show that the set \(A = \{(x, y) : xy > 1\}\) is open. Indeed, this set is given by a strict inequality with continuous left hand side, so the set \(A\) is open. I claim that \(A = E^c\). To prove this I have to verify that each point \(p = (x, y)\) such that \(xy = 1\) is a “bad” point of \(E\), i.e. there is a sequence \(x_k \in E^c\) such that \(\lim_{k \to \infty} x_k = p\). Let \(p = (x, y) = (1 - \frac{1}{k})(x, y)\). Then

$$\lim_{k \to \infty} p_k = \lim_{k \to \infty} (1 - \frac{1}{k})(x, y) = (x, y).$$

On the other hand, \(x_k y_k = (1 - \frac{1}{k})^2 < 1\) for each \(k \in \mathbb{N}\). Hence \(p_k \notin E\) for all \(k\).

6. [10 points] Determine if the set \(E = \mathbb{R}^2 \setminus \{(x, y) : y \neq x^2\}\) is connected.
Solution. I claim that the set $E$ is not connected. Let $U = \{(x, y) : y > x^2\}$, $V = \{(x, y) : y < x^2\}$. Then $U \cap V = E$, both $U$ and $V$ are open (since they are given by strict inequalities with continuous left hand side) and they are both nonempty. Hence $\{U, V\}$ is a separation of $E$, so $E$ is not connected.

7. [15 points] Suppose that $V \subset \mathbb{R}^n$, $f : V \to \mathbb{R}^2$ is continuous and the image of $f$ is the set $E = \{(x, y) : xy = 1\}$. Determine if the set $V$ is compact. (If you claim any property about $E$ you should prove this property.)

Solution. First, note that $E$ contains points $p_k = (k, \frac{1}{k})$, which form an unbounded sequence. Thus $E$ is not bounded, hence $E$ is not compact. Recall that image of a compact set under a continuous function is again compact. Since $V$ is not compact, $E$ is noncompact as well. \Box

8. [10 points] Which of the following are true and which are false (you do not have to give a proof) for a function $f : \mathbb{R}^n \to \mathbb{R}^m$:

(a) If $f$ is differentiable then $f$ is continuous.

(b) If $f$ is not differentiable at some $a \in \mathbb{R}^n$ then some partial derivatives $\frac{\partial f}{\partial x_j}$ do not exist.

(c) If for some $i, j$, the partial derivative $\frac{\partial f}{\partial x_j}$ exists but is not continuous then $f$ is not differentiable.

Solution. (a) is true. (b) and (c) are false. \Box