MATHEMATICS 3220-1. Final Exam (Sample): solutions.

May 6, 2002

The exam is “closed book, closed notes”. All problems should be treated as problems about “proofs”; just the correct computation without proper justification can result in a very low score on the problem.

1. [10 points] Using the definition of the limit of a sequence prove that the following sequence converges:

\[ x_k = \left( \frac{k}{k^2 - 1}, \frac{(-1)^k}{k+1} \right). \]

Solution. I claim that the limit of this sequence is \((0, 0)\). To prove this, given \( \epsilon > 0 \) we have to find \( n_0 \geq 2 \) such that for all \( k \geq n_0 \) we have:
\[
\frac{k}{k^2 - 1} < \epsilon, \quad \frac{1}{k+1} < \epsilon.
\]
Solving the second inequality we get:
\[
\frac{1}{\epsilon} < k + 1 \iff \frac{1}{\epsilon} - 1 < k.
\]
Solving the first inequality we get:
\[
\frac{k}{\epsilon} < k^2 - 1 \iff k^2 - \frac{k}{\epsilon} - 1 > 0,
\]
the latter is satisfied for
\[
k > \frac{1}{2}(\frac{1}{\epsilon} + \sqrt{\frac{1}{\epsilon^2} + 4}).
\]
Let
\[
M = \max\left(\frac{1}{\epsilon} - 1, \frac{1}{2}(\frac{1}{\epsilon} + \sqrt{\frac{1}{\epsilon^2} + 4})\right),
\]
then \( n_0 = \lceil M \rceil + 2 \) will do the job. \( \Box \)

2. [15 points] Compute the following limit or show that it does not exist:

\[
\lim_{(x,y) \to (0,0)} \frac{y^4}{x^2 + y^2}
\]

(you can use limit theorems).

Solution
\[
\frac{y^4}{x^2 + y^2} = y^2 \frac{y^2}{x^2 + y^2}.
\]
Since \( \frac{y^2}{x^2 + y^2} \leq 1 \), we get:
\[
0 \leq \frac{y^4}{x^2 + y^2} \leq y^2.
\]
Since \( \lim_{(x,y) \to (0,0)} y^2 = 0 \), by squeeze lemma we get:

\[
\lim_{(x,y) \to (0,0)} \frac{y^4}{x^2 + y^2} = 0. \quad \blacksquare
\]

3. [15 points] State the definition of a convex set and prove that the set

\[
E = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}
\]
is convex.
Solution. See the textbook.

4. [10 points] State the implicit function theorem.
Solution. See the textbook.

5. [15 points] Prove the theorem on sequential compactness of closed and bounded subsets of \( \mathbb{R}^n \). (Prove the theorem only the direction “If \( E \) is closed and bounded then...”.)
Solution. See the textbook.

6. [10 points] Prove that the function

\[
f(x,y) = \begin{cases} 
\frac{x}{y}, & \text{if } y \neq 0 \\
0, & \text{if } y = 0
\end{cases}
\]
is not differentiable at \((0,0)\).
Solution. I claim that \( f \) is not even continuous at zero: limit along the line \( x = y \) gives us:

\[
\lim_{x \to 0} \frac{x}{x} = 1 \neq f(0,0) = 0.
\]
Since \( f \) is not continuous at zero, it is also non-differentiable at zero.

7. [15 points] Let \( f(x,y) = (x,y^2) \). Using the definition of total derivative verify that

\[
Df(x,y) = L = \begin{bmatrix} 1 & 0 \\
0 & 2y \end{bmatrix}.
\]
Solution. We have to show that

\[
\lim_{h \to 0} \frac{f(x + h_1, y + h_2) - f(x,y) - L(h)}{||h||} = 0.
\]
Considering the numerator we get:

\[
(x + h_1, (y + h_2)^2) - (x,y^2) - (h_1,2yh_2) = (0,h_2^2).
\]
So, we have to prove that

\[
\lim_{(h_1, h_2) \to (0,0)} \frac{h_2^2}{\sqrt{h_1^2 + h_2^2}} = 0.
\]
We have:

\[
\lim_{(h_1, h_2) \to (0,0)} \sqrt{\frac{h_2^2}{h_1^2 + h_2^2}} = \lim_{(h_1, h_2) \to (0,0)} \sqrt{\frac{h_2^4}{h_1^2 + h_2^2}} = 0,
\]
by Problem \# 2. \( \blacksquare \)

8. [10 points] Let \( E = \{(x,y) : y \geq 0\} \). Determine if the subset \( A \subset E \),
\( A = \{(x,0) : -2 \leq x \leq 2\} \), is relatively closed, relatively open or neither.
Solution. 1. I claim that $A$ is relatively closed. Indeed, $A \subset \mathbb{R}^2$ is the intersection

$$A = \{(x, y) : y = 0\} \cap \{(x, y) : -2 \leq x \leq 2\}$$

The first set is given by an equation with continuous left hand side and the second set is given by 2 nonstrict inequalities with continuous function. Hence $A$ is the intersection of two closed sets, hence it is closed. Since $A = A \cap E$, $A$ is relatively closed in $E$ as well.

2. I claim that $A$ is not relatively open in $E$. Indeed, consider $x_k = (0, \frac{1}{k}) \in E \setminus A$, $\lim_{k \to \infty} x_k = (0,0) \in A$. Thus $A$ is not relatively open. $\square$