MATHEMATICS 3220. Homework # 2: Solutions.

1. §8.2, # 1(b) [10 points]. Using Definition 8.11(i) (the definition of limit of a sequence of vectors) prove that the following limit exists:

$$\lim_{k \to \infty} x_k, x_k = \left( \frac{k}{k + 1}, \sin(1/k) \right).$$

Solution. We first will find the limit using limit theorems and then will prove this limit using the definition. To compute the limit of a sequence it suffices to compute the limit of each coordinate sequence (theorem 8.12):

$$\lim_{k \to \infty} \frac{k}{k + 1} = \lim_{k \to \infty} \frac{1}{1 + \frac{1}{k}} = 1,$$

since \( \lim_{k \to \infty} \frac{1}{k} = 0 \). Since \( \sin(t) \) is a continuous function and \( \lim_{k \to \infty} \frac{1}{k} = 0 \) we get:

$$\lim_{k \to \infty} \sin(1/k) = \sin(0) = 0.$$

Thus

$$\lim_{k \to \infty} x_k = (1, 0).$$

We now work our way backwards, namely, given \( \epsilon > 0 \) we will find \( k_0 \geq 0 \) so that for all integers \( k > k_0 \) we have

$$\|x_k\| < \epsilon.$$

Recall that \( \|x_k\| \leq \sqrt{2}\|x_k\|_\infty \), hence it suffices to find \( k_0 \) such that

$$\max\left(\frac{k}{k + 1}, \sin(1/k)\right) < \epsilon/\sqrt{2} \quad \text{for all integers} \quad k > k_0.$$

Solving the inequality \( \frac{k}{k + 1} < \epsilon/\sqrt{2} \) we get:

$$k < (k + 1)\epsilon/\sqrt{2} \iff k(1 - \epsilon/\sqrt{2}) < \epsilon/\sqrt{2}$$

This latter is always true for \( 1 - \epsilon/\sqrt{2} \leq 0 \); for \( 1 - \epsilon/\sqrt{2} > 0 \) we take:

$$k_1 = \frac{\epsilon/\sqrt{2}}{1 - \epsilon/\sqrt{2}}.$$

Now consider the second coordinate. We want to find \( k_0 > 0 \) such that for all integers \( k > k_0 \) we have:

$$0 < \sin\left(\frac{1}{k}\right) < \epsilon/\sqrt{2} \iff$$

$$\frac{1}{k} < \arcsin\left(\frac{\epsilon}{\sqrt{2}}\right) \iff$$

$$k > \frac{1}{\arcsin\left(\frac{\epsilon}{\sqrt{2}}\right)}.$$

Thus \( k_2 = \frac{1}{\arcsin\left(\frac{\epsilon}{\sqrt{2}}\right)} \). Since we want

$$\max\left(\frac{k}{k + 1}, \sin(1/k)\right) < \epsilon/\sqrt{2}, \forall k > k_0,$$
we take
\[ k_0 = \max\left( \frac{\epsilon/\sqrt{2}}{1 - \epsilon/\sqrt{2}}, \frac{1}{\arcsin(\epsilon/\sqrt{2})} \right). \]

2. \# 3. [10 points] Suppose that \( x_k \to 0 \) and \( y_k \) is a bounded sequence (both are in \( \mathbb{R}^n \)). Prove that \( \lim x_k \cdot y_k = 0 \).

Proof. It suffices to prove that
\[
\lim |x_k \cdot y_k| = 0.
\]
Recall that by Cauchy-Schwarz inequality,
\[
|x_k \cdot y_k| \leq \|x_k\| \cdot \|y_k\|.
\]
Now, \( \lim \|x_k\| = 0 \) and the sequence \( \|y_k\| \) is bounded. Hence (by a limit theorem for sequences of real numbers)
\[
\lim \|x_k\| \cdot \|y_k\| = 0.
\]
By the squeeze theorem,
\[
\lim |x_k \cdot y_k| = 0 \]
as well. \( \Box \)

3. \# 5(a). Prove Theorem 8.14 (i) and (ii). [10 points]
(i) prove that a sequence \( (x_k) \) in \( \mathbb{R}^n \) can have at most one limit.

Proof. Suppose that \( (x_k) \) converges to \( a \) and to \( b \). Then for each \( i = 1, ..., n \) we have the convergence of coordinate sequences:
\[
\lim_k x_{ki} = a_i, \lim_k x_{ki} = b_i.
\]
Since a sequence of real numbers has unique limit, we get:
\[
a_i = b_i, i = 1, ..., n.
\]
Hence the vectors \( a \) and \( b \) have the same coordinates, hence \( a = b \). Thus the limit is unique. \( \Box \)

(ii) Suppose that a sequence \( (x_k) \) in \( \mathbb{R}^n \) converges to \( a \). Prove that each subsequence \( (x_{kj}) \) also converges to \( a \).

Proof. Let \( x_k(i), x_{kj}(i), a(i) \) denote the \( i \)-th coordinates of the corresponding vectors. Then for each \( i = 1, ..., n \), \( (x_{kj}(i))_j \) is a subsequence in \( (x_k(i))_k \). Thus by theorem 2.6,
\[
\lim_j x_{kj}(i) = \lim_k x_k(i) = a(i).
\]
Hence
\[
\lim_j x_{kj} = a. \quad \Box
\]