MATHEMATICS 3220. Homework # 10: Solutions.

§11.4, # 1. [10 points] Let \( f : \mathbb{R}^n \to \mathbb{R} \). Suppose that for each unit vector \( u \in \mathbb{R}^n \) the directional derivative \( D_u f(a + tu) \) exists for all \( t \in [0, 1] \). Prove that for each unit vector \( u \) there exists \( t \in (0, 1) \) such that

\[
f(a + u) - f(a) = D_u f(a + tu).
\]

Solution. For the given unit vector \( u \) consider the function \( h(t) = f(a + tu) \), \( h : \mathbb{R} \to \mathbb{R} \). The domain of this function is an interval \((-\delta, 1 + \delta)\) for some \( \delta > 0 \). The derivative of this function exists for all \( t \in [0, 1] \) because it equals

\[
h'(t) = \lim_{s \to 0} \frac{f(a + tu + su) - f(a + tu)}{s} = D_u f(a + tu).
\]

Now, \( f(a + u) - f(a) = h(1) - h(0) \). By the Mean Value Theorem for the functions of 1 variable we get: There exists \( t \in (0, 1) \) such that

\[
f(a + u) - f(a) = h(1) - h(0) = h'(t)(1 - 0) = D_u f(a + tu). \quad \square
\]

§11.4, # 3(b). [10 points] Write Taylor’s formula for \( f(x, y) = \sqrt{x} + \sqrt{y} \), \( a = (1, 4) \) and \( p = 3 \).

Solution. Let’s compute the partial derivatives of the order \( \leq p = 3 \) for the function \( f(x, y) \):

\[
f_x = \frac{1}{2} x^{-1/2}, \quad f_y = \frac{1}{2} y^{-1/2}.
\]

Hence all higher order mixed derivatives are zero. The only nonzero derivatives are:

\[
f_{xx} = -\frac{1}{4} x^{-3/2}, \quad f_{yy} = -\frac{1}{4} y^{-3/2},
\]

\[
f_{xxx} = \frac{3}{8} x^{-5/2}, \quad f_{yyy} = \frac{3}{8} y^{-5/2}.
\]

Substituting \( a = (1, 4) \) we get:

\[
f(a) = 1 + 2 = 3, \quad f_x(a) = \frac{1}{2}, \quad f_{xx}(a) = -\frac{1}{4};
\]

\[
f_y(a) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}, \quad f_{yy}(a) = -\frac{1}{4} \cdot \frac{1}{8} = -\frac{1}{32}.
\]

Hence for a vector \( h = (h_1, h_2) = (x, y) - (1, 4) = (x - 1, y - 4) \) the Taylor’s formula is:

\[
f(x, y) = f(1, 4) + \sum_{k=1}^{2} \frac{1}{k!} D^{(k)} f(a; h) + \frac{1}{3!} D^{(3)} f(c; h)
\]

for a certain \( c = (c, d) \in [a, x] \). That is:

\[
f(x, y) = 3 + f_x(a)(x - 1) + f_y(a)(y - 4) + \frac{1}{2}[(x - 1)^2 f_{xx}(a) + (y - 4)^2 f_{yy}(a)] + \\
\quad + \frac{1}{6}[(x - 1)^3 f_{xxx}(c) + (y - 4)^3 f_{yyy}(c)] =
\]
\[ f(x, y) = 3 + \frac{1}{2}(x - 1) + \frac{1}{4}(y - 4) + \frac{1}{2}[\frac{-1}{4}(x - 1)^2 + \frac{-1}{32}(y - 4)^2] + \frac{1}{16}(x - 1)^3e^{-5/2} + (y - 4)^3d^{-5/2}. \]

\[ \Box \]

§11.4, # 3(c). [10 points] Write Taylor’s formula for \( f(x, y) = e^{xy}, \ a = (0, 0) \) and \( p = 3 \).
Solution. Let’s compute the partial derivatives of the order \( \leq p = 3 \) for the function \( f(x, y) \):

\[
\begin{align*}
  f_x &= ye^{xy}, f_y = xe^{xy}; \\
  f_{xx} &= y^2e^{xy}, f_{yy} = x^2e^{xy}, f_{xy} = e^{xy} + xy e^{xy}; \\
  f_{xxx} &= y^3e^{xy}, f_{xxy} = 2ye^{xy} + y^2xe^{xy}, f_{yyx} = 2xe^{xy} + x^2ye^{xy}, f_{yyy} = x^3e^{xy}
\end{align*}
\]

Note that almost all of these derivatives vanish at \((0,0)\), the only nonzero derivative is:

\[ f_{xy}(0, 0) = 1. \]

Hence for a vector \( \mathbf{h} = (x, y) \) the Taylor’s formula at \((0,0)\) is:

\[ f(x, y) = f(0, 0) + \sum_{k=1}^{2} \frac{1}{k!} D^{(k)}f((0,0); \mathbf{h}) + \frac{1}{3!} D^{(3)}f(\mathbf{c}; \mathbf{h}) \]

for a certain \( \mathbf{c} = (c, d) \in [0, x] \). That is:

\[
\begin{align*}
  f(x, y) &= 1 + \frac{1}{2}(xy f_{xy}(0, 0) + xy f_{yx}(0, 0)) + \\
  &+ \frac{1}{6}[x^3d^3e^{cd} + 3 \cdot x^2y(2de^{cd} + d^2ce^{cd}) + 3 \cdot y^2x(2ce^{cd} + c^2de^{cd}) + y^3c^3e^{cd}] = \\
  &= 1 + xy + \frac{e^{cd}}{6}[x^3d^3 + 3 \cdot x^2yd(2 + dc) + 3 \cdot y^2xc(2 + cd) + y^3c^3].
\end{align*}
\]

Note that we get the factor 3 in this formula because we have to take \( f_{xxy} = f_{yxx} = f_{xyx} \) (three equal derivatives) as well as \( f_{yyx} = f_{yx} = f_{xyy} \) (three equal derivatives).

\[ \Box \]