MATH 2270-2. Final Test: Solutions.

**Problem 1.** (10 points) Find inverse of the matrix:

\[ A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \]

Use any method you like.

Solution.

\[ A^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix}. \]

**Problem 2.** (15 points) Let \( V \) be the linear space of continuous functions of one variable \( f : \mathbb{R} \to \mathbb{R} \) (with the usual operations of sum and multiplication by scalars). Let \( Z \) be the subset in \( V \) which consists of all continuous functions \( f \) which have integer values at zero (i.e., we allow \( f(0) \) to be \( 0, \pm 1, \pm 2, \ldots \), but for instance \( f(0) = 0.5 \) is not allowed). Determine whether or not the subset \( Z \) is a subspace in \( V \). Justify your answer!

Solution. \( Z \) is not a subspace. For instance, take function \( f(x) = 1 \), \( \alpha = 0.5 \). Then \( \alpha f(0) = 0.5 \) is not an integer, i.e. \( \alpha f(x) \) does not belong to \( Z \).

**Problem 3.** (15 points) 3. [15 points] Using standard basis in \( P_2 \) find matrix representation, rank, nullity and basis of the image of the linear transformation \( T : P_2 \to P_2 \) which is given by the formula:

\[ T(p(x)) = xp'(x) + x^2p(1). \]

Here \( p' \) denotes the derivative.

Solution. \( T(1) = 0 + x^2 \), has coordinates \( (0, 0, 1) \). Next, \( T(x) = x + x^2 \), has coordinates \( (0, 1, 1) \). Lastly, \( T(x^2) = 2x^2 + x^2 = 3x^2 \) has coordinates \( (0, 0, 3) \). Hence the matrix of \( T \) is

\[ A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}. \]

This matrix has rank 2 and nullity 1. The basis of the image consists of the vectors \( (0, 0, 1) \) and \( (0, 1, 1) \). Converting these vectors into polynomials we get: basis of the image of \( T \) is \( \{x^2, x + x^2\} \). \( \square \)

**Problem 4.** (10 points) Write the augmented matrix and find all solutions of the linear system:

\[ \begin{align*} x_1 + x_2 + x_3 - 2x_4 &= 3 \\
2x_1 + x_2 + 3x_3 + 2x_4 &= 5 \\
3x_1 + 2x_2 + 4x_3 &= 8 \end{align*} \]
Solution. \(x_1 = 2 - 2t - 4s, x_2 = 1 + t + 6s, x_3 = t, x_4 = s\) are the parameters.

**Problem 5.** (15 points) Find an orthonormal basis in the subspace \(V\) in \(\mathbb{R}^4\) spanned by the vectors

\[
\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ -4 \\ 6 \end{pmatrix}.
\]

Solution. \(\vec{u}_1 = \frac{1}{2}(1, 1, -1, 1), \vec{u}_2 = \frac{1}{2}(-1, 1, 1, 1)\).

**Problem 6.** (15 points) Find all eigenvalues of the matrix:

\[
\begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}
\]

For each eigenvalue find a basis of the corresponding eigenspace. Determine if the matrix is diagonalizable.

Solution. The first eigenvalue is 3, the basis of \(E_3\) is \((0, 0, 1)\). The second eigenvalue is 2, the basis of \(E_2\) is \((0, -1, 1)\). Matrix is not diagonalizable since the sum of dimensions of eigenspaces is 2 and not 3.

**Problem 7.** (10 points) Compute the following determinant using the definition of the determinant (i.e. identify the patterns with the nonzero product of the entries, compute the \(\pm\) signs, etc.):

\[
\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.
\]

Solution. There is only one pattern with nonzero product, it is the product of the entries equal to 1. The pattern contains 2 inversions, hence the determinant equals \((-1)^2 = 1\).

**Problem 8.** (10 points) Consider the vector space \(V = \text{Span}\{\sin^2(t), \cos^2(t), t\}\) with the basis: \(T = \{\sin^2(t), 1, t - 1\}\). Compute the coordinates of the function \(2(\sin^2(t) + \cos^2(t)) - t\) with respect to the basis \(T\).

Solution. \(2(\sin^2(t) + \cos^2(t)) - t = 0 \cdot \sin^2(t) + 1 \cdot 1 + (-1) \cdot (t - 1).\) Hence the coordinates are \((0, 1, -1)\).