Problem 1. [15 points] Suppose that $A$ is an invertible matrix $n \times n$ with the eigenvalues $\lambda_1, \ldots, \lambda_k$. What are eigenvalues of $A^{-1}$? Justify your answer.

Solution. First of all, since $A$ is invertible, all its eigenvalues are nonzero. For each $i$ there exists a nonzero vector $v_i$ such that 

$$Av_i = \lambda_i v_i.$$ 

By multiplying this equation by $A^{-1}$ and by $\lambda_i^{-1}$ we get:

$$\frac{1}{\lambda_i} v_i = A^{-1} v_i.$$ 

Hence $\frac{1}{\lambda_i}$ is an eigenvalue of $A^{-1}$ for each $i$. The same logic shows that these are the only eigenvalues of $A^{-1}$. Hence the eigenvalues of $A^{-1}$ are $\lambda_1^{-1}, \ldots, \lambda_k^{-1}$. \hfill $\square$

Problem 2. (20 points) Compute eigenvalues and bases of the corresponding eigenspaces for the matrix 

$$A = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$$

Using this, compute the limit $\lim_{n \to \infty} A^n$.

Solution.

$$A = \frac{1}{4} \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}.$$ 

Let’s diagonalize the matrix 

$$B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}.$$ 

The eigenvalues of $B$ are the roots of the polynomial $(\lambda - 2)(\lambda - 3) - 2 = \lambda^2 - 5\lambda + 4$. Hence the eigenvalues of $B$ are 1, 4. Thus the eigenvalues of $A$ are $1/4, 1$. Now let’s find the eigenvectors of $B$: 

$E_1$ is the kernel of 

$$\begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix},$$

thus the basis of $E_1$ is the vector $v_1 = (1, -1)$.

$E_4$ is the kernel of 

$$\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix},$$

thus the basis of $E_4$ is the vector $v_1 = (1, 2)$. Hence the eigenbasis of $B$ (and hence of $A$) is: 

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$ 

Therefore 

$$S = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, S^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix},$$

$$B = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 4 \end{bmatrix}.$$
hence
\[ A = \frac{1}{4} B = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}. \]

Hence
\[ A^n = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1/(4^n) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}. \]

The limit \( \lim_{n \to \infty} 1/(4^n) \) is zero. Hence
\[ \lim_{n \to \infty} A^n = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}. \]

**Problem 3.** [15 points] Using standard basis in \( P_2 \) find matrix representation, and eigenvalues of the linear transformation \( T : P_2 \to P_2 \) which is given by the formula:
\[ T(p(x)) = p(x + 1) - p'(x). \]

**Solution.** \( T(1) = 1 - (1)' = 1 \), has coordinates \((1, 0, 0)\). \( T(x) = x + 1 \), has coordinates \((1, 0, 0)\). \( T(x^2) = (x + 1)^2 - (x^2)' = x^2 + 2x + 1 - 2x = 1 + x^2 \), the latter has coordinates \((1, 0, 1)\). Thus the matrix of \( T \) is
\[ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

This matrix is upper triangular, hence its eigenvalues are its diagonal entries. Thus \( A \) has only one eigenvalue: \( \lambda = 1 \).

**Problem 4.** [15 points] Determine if the quadratic form \( q \) is positive definite:
\[ q(x, y) = x^2 + 3y^2 + 4xy. \]

**Solution.** The matrix of this quadratic form is
\[ A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}. \]

The determinant of this matrix is \( 3 - 4 = -1 < 0 \), hence \( A \) is not positive-definite. Therefore \( q \) is not positive definite.

**Problem 5.** [15 points] Is the following set \( S \) a linear subspace in \( P_2 \)? Justify your answer. Find a basis of \( S \) if \( S \) is linear.
\[ S = \{ p(x) : p(0) = p(1) \}. \]

**Solution.** First, let’s check that \( S \) is a subspace:
(a) \( p(x) = 0 \) is in \( S \), since \( 0 = p(0) = p(1) = 0 \).
(b) Suppose that \( p(x), q(x) \) are in \( S \). Thus \( p(0) = p(1), q(0) = q(1) \). It follows that \( (p + q)(0) = p(0) + q(0) = p(1) + q(1) = (p + q)(1) \). Thus \( p + q \) is in \( S \).
(c) Suppose that \( \alpha \) is a scalar, \( p(x) \) is a polynomial in \( S \). Thus \( p(0) = p(1) \). It follows that \( \alpha p(0) = \alpha p(1) \). Hence \( \alpha p(x) \) is in \( S \).

Therefore \( S \) is a subspace.
Second, let’s find the general polynomial in $S$:

$$p(x) = a_0 + a_1 x + a_2 x^2, \ p(0) = p(1) \Rightarrow \ a_0 = a_0 + a_1 + a_2,$$

hence $a_2 = -a_1$, and $a_0, a_1$ are arbitrary parameters. Thus to find a basis we first let $a_0 = 1, a_1 = 0$, hence we get $p(x) = 1$. Next, we take $a_0 = 0, a_1 = 1$, hence we get $q(x) = x - x^2$. Therefore $\{1, x - x^2\}$ is a basis.

**Problem 6.** (20 Points) Determine if the following matrices are similar:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$ 

Justify your answer.

Solution. Note that $A$ and $B$ have same eigenvalues of the same algebraic multiplicity: 1 is an eigenvalue of algebraic multiplicity 2 and 2 is an eigenvalue of algebraic multiplicity 1. However this is not enough for similarity of the matrices. Let’s compute geometric multiplicities of the eigenvalue 1.

For the matrix $A$ the subspace $E_1$ is the kernel of the matrix

$$I - A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

The latter has rank 2 and nullity 1. Therefore the geometric multiplicity of 1 in this case equals $\dim(E_1) = 1$.

For the matrix $B$ the subspace $E_1$ is the kernel of the matrix

$$I - B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

The latter has rank 1 and nullity 2. Therefore the geometric multiplicity of 1 in this case equals $\dim(E_1) = 2$. For similar matrices geometric multiplicities of eigenvalues should be the same, therefore $A, B$ are not similar.

Alternative solution. Check that $B$ is diagonalizable and $A$ is not. Therefore these matrices cannot be similar. 

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\square
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