MATHEMATICS 2270. Homework # 10: Solutions.

Total= 40 points.

1. § 7.1, # 6. [5 points] If \( \mathbf{v} \) is an eigenvector of both \( A \) and \( B \), is \( \mathbf{v} \) necessarily an eigenvector of \( AB \)?

Solution. We have \( A \mathbf{v} = \lambda \mathbf{v}, \ B \mathbf{v} = \mu \mathbf{v}, \) hence

\[
A(B \mathbf{v}) = A \mu \mathbf{v} = \mu A \mathbf{v} = (\mu \lambda) \mathbf{v}.
\]

Hence \( \mathbf{v} \) is an eigenvector of \( AB \). \( \square \)

2. § 7.1, # 34. [10 points] Find a \( 2 \times 2 \) matrix \( A \) such that \( \mathbf{v} = (3, 1) \) and \( \mathbf{u} = (1, 2) \) are its eigenvectors with eigenvalues 5 and 10 respectively. [Hint: first write down the matrix of the corresponding linear transformation using the coordinates associated with the basis of eigenvectors and then do the change of coordinates.]

Solution. Let \( T(\mathbf{x}) = A \mathbf{x} \). We start with the basis \( \mathcal{B} = \{ (3, 1), (1, 2) \} \), with respect to this basis the transformation \( T \) is given by the matrix:

\[
B = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix},
\]

since \( T(\mathbf{v}) = 5 \mathbf{v}, T(\mathbf{u}) = 10 \mathbf{u} \). The transition matrix from \( \mathcal{B} \)-coordinates to the standard coordinates is

\[
S = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix},
\]

whose inverse is

\[
S^{-1} = \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.
\]

Therefore

\[
A = SBS^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} =
\]

\[
= \begin{bmatrix} 4 & 3 \\ -2 & 11 \end{bmatrix}.
\]

3. § 7.2, # 8. [5 points] Use characteristic polynomial to find eigenvalues of the matrix \( A \) and their algebraic multiplicities:

\[
A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}.
\]

Solution. The characteristic polynomial equals

\[
f_A(\lambda) = \begin{vmatrix} \lambda + 1 & 1 & 1 \\ 1 & \lambda + 1 & 1 \\ 1 & 1 & \lambda + 1 \end{vmatrix} = (\lambda + 1)^3 + 2 - 3(\lambda + 1) = \lambda^3 + 3\lambda^2 = \lambda^2(\lambda + 3).
\]

Therefore \( \lambda = 0 \) is an eigenvalue of algebraic multiplicity 2 and \( \lambda = -3 \) is an eigenvalue of algebraic multiplicity 1. \( \square \)
4. §7.2, # 12. [10 points] Use characteristic polynomial to find eigenvalues of the matrix $A$ and their algebraic multiplicities:

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 2 & -3 \end{bmatrix}.$$ 

Solution. The characteristic polynomial equals

$$f_A(\lambda) = \begin{vmatrix} \lambda - 2 & 1 & 0 & 0 \\ 1 & \lambda + 1 & 0 & 0 \\ 0 & 0 & \lambda - 3 & 4 \\ 0 & 0 & -2 & \lambda + 3 \end{vmatrix} = ((\lambda - 2)(\lambda + 1) - 1)((\lambda - 3)(\lambda + 3) + 8) = [\lambda^2 - \lambda - 3][\lambda^2 - 1].$$

Thus, $1, -1, \frac{1}{2}(1 + \sqrt{13})$ and $\frac{1}{2}(1 - \sqrt{13})$ are eigenvalues of algebraic multiplicity 1.

5. §7.3, # 12. [5 points] Find all real eigenvalues of $A$, find basis of each eigenspace and if possible find an eigenbasis.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

Solution. The characteristic polynomial equals $(\lambda - 1)^3$, since the matrix is upper triangular. Thus $\lambda = 1$ is the only eigenvalue. To find the corresponding eigenspace $E_1$ consider the equation

$$(I - A)x = 0,$$

which has the augmented matrix

$$\begin{bmatrix} 0 & -1 & 0 & | & 0 \\ 0 & 0 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Thus

$$x = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix},$$

where $x_1$ is a parameter. Hence the basis in $E_1$ consists of the single vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$ 

In particular, there is no eigenbasis.

6. §7.3, # 36. [5 points] Are the following matrices similar:

$$A = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}?$$

Solution. Traces of similar matrices should be the same. However $tr(A) = 3$, $tr(B) = 4$ are different. Hence $A$ is not similar to $B$. (Note that we cannot use the determinants: both matrices have the same determinant $-5$.)