Math 3210-2. 3-rd Midterm Test: Solutions.

1. (15 points) State the definition of uniformly continuous function. Give example of a function which is continuous but not uniformly continuous. Justify the example!

Solution. \( f \) is uniformly continuous on a set \( S \) if for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( x, y \in S \) satisfy \( |x - y| < \delta \) then \( |f(x) - f(y)| < \epsilon \). Consider the function \( f(x) = \frac{1}{x} \) on \((0, \infty)\). Then \( f \) is continuous (since each rational function is continuous on its domain). Let \( x_n = \frac{1}{n} \in (0, \infty) \). Then \( (x_n) \) is Cauchy since it converges to zero. On the other hand, \( f(x_n) = n \) diverges to \(+\infty\), hence it is not Cauchy. Since for each absolutely continuous function \( f \), the \( f(x_n) \) is Cauchy provided that \( x_n \) is Cauchy, we conclude that \( f(x) = \frac{1}{x} \) is not Cauchy. \( \square \)

2. (20 points) Prove that if a function \( f \) is differentiable at \( a \) then \( f \) is continuous at \( a \).

Solution. See the textbook.

3. (15 points) Show that \( e^x > x \) for all real numbers \( x \geq 0 \).

Solution. \( (e^x)' = e^x > 0 \). Hence the function \( e^x \) is strictly increasing on \([0, \infty)\).

Consider the function \( f(x) = e^x - x \). Then \( f(0) = 1 - 0 = 1 > 0 \).
Let’s prove that \( f \) is increasing on \([0, \infty)\). For \( x > 0 \), \( f'(x) = e^x - 1 > 1 - 1 = 0 \). Hence \( f'(x) > 0 \) on \((0, \infty)\) and so the function \( f \) is strictly increasing. Thus \( f(x) > f(0) > 0 \) for each \( x \in (0, \infty) \). Hence \( 3^x > x \) for all real numbers \( x \geq 0 \). \( \square \)

4. (15 points) Determine if the limit

\[
\lim_{x \to -\infty} x(\sin(x) + 2)
\]

exists and compute this limit if it exists. You can use limit theorems if you like.

Solution. Let \( x_n \to -\infty \) be a sequence of real numbers. Recall that \( \sin(x) \geq -1 \) for all \( x \), hence \( \sin(x) + 2 \geq 1 \) for all \( x \). Thus

\[
x_n(\sin(x_n) + 2) \geq x_n.
\]

Hence (by sandwich lemma)

\[
\lim_{n \to \infty} x_n(\sin(x_n) + 2) \geq \lim_{n \to \infty} x_n = -\infty.
\]

Therefore

\[
\lim_{x \to -\infty} x(\sin(x) + 2) = -\infty. \quad \square
\]
5. (15 points) Determine if the function

$$f(x) = \begin{cases} 
  x^2 \cos(1/x) & \text{if } x > 0 \\
  x^3 & \text{if } x \leq 0 
\end{cases}$$

is differentiable at zero and compute the derivative if it exists. You can use limit theorems if you like.

Solution. Consider

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f(x)}{x}.$$ 

To compute this limit first consider

$$\lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0^+} \frac{x^2 \cos(1/x)}{x} = \lim_{x \to 0^+} x \cos(1/x) = 0$$

since $\cos(1/x)$ is bounded and $\lim_{x \to 0^+} x = 0.$

Now consider

$$\lim_{x \to 0^-} \frac{f(x)}{x} = \lim_{x \to 0^-} \frac{x^3}{x} = \lim_{x \to 0^-} x^2 = 0.$$ 

Hence the limit from the left equals the limit from the right, thus $f'(0)$ exists and

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = 0. \quad \square$$
6. (20 points) Show that the function $f : [0, \infty) \to [0, \infty)$, $f(x) = x^2$, admits a continuous inverse which is defined on the whole interval $[0, \infty)$. (You can use any theorem about continuous functions you like.)

Solution. The function $x^2$ is continuous, hence the image of $f$ is an interval $J$. Let’s find the lower and upper bounds of this interval. $f(0) = 0 \leq x^2$ for each $x$, hence $0 \in J$ is the lower bound for $J$.

$$\lim_{x \to \infty} x^2 = +\infty,$$

hence the upper bounds for $J$ is $+\infty$. Hence $J = [0, \infty)$.

The function $f$ is strictly increasing on $[0, \infty)$ since for $y > x \geq 0$ we have $y^2 > x^2$. Thus (by the inverse function theorem) $f$ admits a continuous inverse function defined on whole interval $[0, \infty)$. 
