MATHEMATICS 3210. Homework #7: Solution.

1. (a) Let \( x_n = \cos(\frac{2n\pi}{n}) \). (b) Find a convergent subsequence and compute \( \limsup x_n \).

Solution. (a) Recall that \( \cos(x + 2n\pi) = \cos(x) \). Thus
\[
x_{6n} = \cos\left(\frac{6n\pi}{3}\right) = \cos(2n\pi) = 1.
\]
This subsequence is constant and converges to 1.
(b) If \( n = 6k + r \) where \( 0 \leq r < 6 \) then
\[
x_{6k+r} = \cos(2k\pi + \frac{r\pi}{3}) = \cos\left(\frac{r\pi}{3}\right).
\]
The latter takes values 1 if \( r = 0 \); 1/2, if \( r = 1 \); -1/2, if \( r = 2 \); 0 if \( r = 3 \); -1/2, if \( r = 4 \); 1/2 if \( r = 5 \). The largest of these numbers equals 1. For each \( N \in \mathbb{N} \) there is a natural number \( k \) such that \( 6k \geq N \).
Hence
\[
\sup\{x_n | n \geq N\} = 1.
\]
Thus
\[
\limsup x_n = \lim_{N \to \infty} \sup\{x_n | n \geq N\} = 1.
\]

2. Determine whether or not the sequence
\[
x_n = \frac{n + n(-1)^n}{n + 1}
\]
contains a convergent subsequence.

Solution. Let’s show that this sequence is bounded:
\[
|x_n| = \frac{|n + n(-1)^n|}{n + 1} \leq \frac{n + n|(-1)^n|}{n + 1} = \frac{n + n}{n + 1} = \frac{2n}{n + 1}.
\]
We would like to show that the sequence \( \frac{2n}{n+1} \) is bounded. One way to prove it is to note that this sequence converges:
\[
\lim \frac{2n}{n + 1} = \lim \frac{2}{1 + 1/n} = 2.
\]
(By the “fraction” limit theorem.) Since each convergent sequence is bounded we see that there exists \( C > 0 \) such that
\[
\frac{2n}{n + 1} \leq C
\]
Thus \( |x_n| \leq \frac{2n}{n+1} \leq C \). So the sequence \( (x_n) \) is bounded. By the Bolzano-Weierstrass theorem, \( (x_n) \) has a convergent subsequence. \( \square \)

14.5 (a). Suppose that \( \sum a_n, \sum b_n \) converge. Prove that \( \sum(a_n + b_n) \) converge and
\[
\sum(a_n + b_n) = \sum a_n + \sum b_n.
\]
Solution. Follows immediately from the “sum” theorem for the limnits of sequences.

14.7. Show that if \( \sum a_n \) converges and \( a_n \geq 0 \) for all \( n \) then for each \( p > 1 \) the series \( \sum a_n^p \) also converges.

Solution. Since \( a_n \) converges, \( \lim a_n = 0 \). Hence there exists \( N \in \mathbb{N} \) such that \( a_n < 1 \) for each \( n \geq N \). For each \( 0 \leq x < 1 \) we have \( x^p \leq x \). Hence for \( n \geq N \) we have:

\[
0 \leq a_n^p \leq a_n.
\]

Thus \( \sum a_n \) converges by the comparison test. \( \square \)

14.1. (a) Determine if the following sequences converges:
\( \sum \frac{n^4}{2^n} \).

Solution. Let’s apply the ratio test to this sequence:

\[
\frac{a_{n+1}}{a_n} = \frac{(n+1)^4 2^n}{2^{n+1} n^4} = \frac{1}{2} \left( \frac{n+1}{n} \right)^4.
\]

By the “sum” limit theorem

\[
\lim \frac{n+1}{n} = \lim (1 + 1/n) = 1.
\]

By the product limit theorem,

\[
\lim \frac{1}{2} \left( \frac{n+1}{n} \right)^4 = \frac{1}{2} 1^4 = \frac{1}{2} < 1.
\]

Since this limit is \( < 1 \), by the ratio test, the series converges.

**Problem.** Determine if the series converges:

\[
\sum 2 + \cos(n) 4^n.
\]

Solution. Let’s apply the fraction test to this sequence:

\[
\frac{a_{n+1}}{a_n} = \frac{4^n (2 + \cos(n+1))}{4^{n+1} (2 + \cos(n))} = \frac{12 + \cos(n+1)}{4} \frac{2 + \cos(n)}{2 + \cos(n)}
\]

We would like to show that

\[
\limsup \frac{12 + \cos(n+1)}{4} \frac{2 + \cos(n)}{2 + \cos(n)} \leq \frac{3}{4} < 1.
\]

Indeed,

\[
\frac{1}{4} \left| \frac{2 + \cos(n+1)}{2 + \cos(n)} \right| = \frac{3}{4}.
\]

Thus \( \limsup \frac{12 + \cos(n+1)}{4} \frac{2 + \cos(n)}{2 + \cos(n)} \leq \frac{3}{4} \). Hence by the ratio test, the series converges. \( \square \)