
December 7, 2001

1. Section 33, # 33.4, 33.5, 33.7, 33.8, 33.13.

# 33.4. Give an example of a function $f$ on $[0, 1]$ that is not integrable for which $|f|$ is integrable.

Solution. Take $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = -x$ if $x \notin \mathbb{Q}$. This function is similar to the Dirichlet’s function, so we have:

$$U(f) = 1/2, L(f) = -1/2.$$ 

Hence $f$ is not integrable. On the other hand, $|f(x)| = x$ which is an integrable function. $\square$

# 33.5. Show that $| \int_{-2\pi}^{2\pi} x^2 \sin^8(e^x) \, dx | \leq \frac{16\pi^3}{3}$.

Solution. Note that $|\sin^8(e^x)| \leq 1$, hence

$$| \int_{-2\pi}^{2\pi} x^2 \sin^8(e^x) \, dx | \leq \int_{-2\pi}^{2\pi} |x^2| |\sin^8(e^x)| \, dx \leq \int_{-2\pi}^{2\pi} x^2 \, dx.$$ 

Since $\int x^2 \, dx = \frac{x^3}{3} + \text{const}$, we have:

$$\int_{-2\pi}^{2\pi} x^2 \, dx = \frac{1}{3}[(2\pi)^3 - (-2\pi)^3] = \frac{2}{3}8\pi^3 = \frac{16\pi^3}{3}. \quad \square$$

# 33.7. Let $f$ be a function on $[a, b]$ so that there exists $B > 0$ such that $|f(x)| \leq B$ for all $x \in [a, b]$.

(a) Show that $U(f^2, P) - L(f^2, P) \leq 2B[U(f, P) - L(f, P)]$ for all partitions $P$ of $[a, b]$.

Solution. For each interval $[t_{k-1}, t_k]$ of the partition $P$ we have:

$$M(f^2, [t_{k-1}, t_k]) = M(f, [t_{k-1}, t_k])^2, m(f^2, [t_{k-1}, t_k]) = m(f, [t_{k-1}, t_k])^2$$

$$U(f^2, P) - L(f^2, P) = \sum_{k=1}^{n} [M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k])](t_k - t_{k-1}) =$$

$$\leq \sum_{k=1}^{n} (t_k - t_{k-1})[M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) \cdot [M(f, [t_{k-1}, t_k]) + m(f, [t_{k-1}, t_k])] \leq$$

$$\sum_{k=1}^{n} (t_k - t_{k-1})2B[M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])] = 2B[U(f, P) - L(f, P)]. \quad \square$$
(b) Show that if \( f \) is integrable on \([a, b]\) then \( f^2 \) is also integrable.

Solution. Since \( f \) is integrable, for each \( \epsilon > 0 \) there is \( \delta > 0 \) so that if \( \text{mesh}(P) < \delta \) then \( U(f, P) - L(f, P) < \frac{\epsilon}{2B} \). Thus (according to (a))

\[
U(f^2, P) - L(f^2, P) \leq 2B[U(f, P) - L(f, P)] \leq 2B \frac{\epsilon}{2B} = \epsilon.
\]

This implies that \( f^2 \) is also integrable. \( \square \)

# 33.8. Let \( f, g \) be integrable on \([a, b]\).

(a) Show that \( fg \) is integrable on \([a, b]\).

Solution. We know that \( f + g \) is integrable, hence \((f + g)^2\) is integrable too (see previous problem). The functions \( f^2, g^2 \) are integrable as well. Thus the function

\[
\frac{1}{2}[(f + g)^2 - f^2 - g^2]
\]

is also integrable. On the other hand,

\[
(f + g)^2 = f^2 + 2fg + g^2, \quad fg = \frac{1}{2}[(f + g)^2 - f^2 - g^2]
\]

Thus \( fg \) is integrable. \( \square \)

# 33.13. Suppose that \( f, g \) are continuous functions on \([a, b]\) such that \( \int_a^b f = \int_a^b g \).

Show that there exists \( x \in [a, b] \) such that \( f(x) = g(x) \).

Solution. Consider the function \( h = f - g \). This function is continuous on \([a, b]\),
\( \int_a^b h = \int_a^b f - \int_a^b g = 0 \). Hence by the mean value theorem for integrals, there exists \( x_0 \in [a, b] \) such that \( h(x_0)(b-a) = \int_a^b h = 0 \). Thus \( h(x_0) = 0 \). Hence \( f(x) = g(x) \). \( \square \)

2. Section 34, # 34.2, 34.3, 34.11.

# 34.2. (a) Calculate

\[
\lim_{x \to 0} \frac{1}{x} \int_0^x e^t \, dt.
\]

Solution. Let \( F(x) = \int_0^x e^t \, dt \). Then \( F'(x) = e^x = f(x) \) for each \( x \). Note that \( F(0) = 0 \). Hence

\[
\lim_{x \to 0} \frac{1}{x} \int_0^x e^t \, dt = \lim_{x \to 0} \frac{1}{x} (F(x) - F(0)) = F'(0) = f(0) = 1.
\]

(b) Similar.

# 34.3. Let \( f \) be defined as follows: \( f(t) = 0 \) for \( t < 0 \) and \( f(t) = t \) for \( 0 \leq t \leq 1 \), \( f(t) = 4 \) for \( t > 0 \).

(a) Compute the function \( F(x) = \int_0^x f(t) \, dt \).

Solution. Case 1: \( x \leq 0 \). Then \( F(x) = \int_0^x f(t) \, dt = \int_0^0 f(t) = 0 \).

Case 2. \( 0 < x \leq 1 \). Then \( F(x) = \int_0^x f(t) \, dt = \int_0^x t \, dt = x^2/2 \).

Case 3. \( 1 < x < \infty \). Then

\[
F(x) = \int_0^1 f(t) \, dt + \int_1^x f(t) \, dt = 0.5 + \int_1^x 4 \, dt = 0.5 + 4(x - 1) = 4x - 3.5.
\]
(b) Is $f$ continuous?
Solution. $F$ is continuous since it is integral of a bounded function (theorem 34.3).
(c) Where is $F$ differentiable? Compute the derivative.
Solution. By inspection, $F$ is differentiable for each $x \neq 0, 4$. The derivative of $F$ on the intervals $(-\infty, 0), (0, 1), (1, \infty)$ is $0, t, 4$ respectively. Let’s verify that $F$ is not differentiable at $0$ (similar argument works for $x = 1$). Consider

$$\lim_{x \to 0^-} \frac{F(x) - F(0)}{x} = \lim_{x \to 0^-} 0 = 0,$$

$$\lim_{x \to 0^+} \frac{F(x) - F(0)}{x} = \lim_{x \to 0^+} 1 = 1.$$

Thus $\lim_{x \to 0} \frac{F(x) - F(0)}{x}$ does not exist, so the function $F$ is not differentiable at zero. The same works at $x = 1$. \qed

# 34.11. Suppose that $f$ is a continuous function on $[a, b]$ and that $f(x) \geq 0$ for all $x \in [a, b]$. Suppose that $\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a, b]$.
Solution. Consider the function $F(x) = \int_a^x f(t)dt$. For $x < y$ we have

$$F(y) - F(x) = \int_x^y f(t)dt \geq 0.$$

Hence $F$ is increasing. On the other hand, $F(a) = 0, F(b) = \int_a^b f(t)dt = 0$. Thus $F(x) = 0$ for each $x \in [a, b]$. Hence, $f(x) = F'(x) = 0$ for each $x \in [a, b]$. \qed