Solutions for the sample of the 2-nd midterm test.

1. Does the following limit exists? Explain your solution.

\[ \lim_{(x,y) \to (0,0)} \frac{2x - y}{x + 3y} \]

**Solution.** The limit does not exist. To verify this claim consider the limit along the x-axis:

\[ \lim_{(x,0) \to (0,0)} \frac{2x - 0}{x + 0} = 2. \]

Now consider the limit along the y-axis:

\[ \lim_{(0,y) \to (0,0)} \frac{0 - y}{0 + 3y} = -1/3. \]

The limits are different, hence the limit of the function of two variables does not exist.

2. Find the Cartesian equation corresponding to the following spherical coordinates equation:

\( (\rho \sin(\phi))^2 = \rho \cos(\phi). \)

**Solution.** \( \rho \cos(\phi) = z, \rho \sin(\phi) = \sqrt{x^2 + y^2}, \) hence the equation becomes:

\[ x^2 + y^2 = z. \]

3. Find equation (in the form \( Ax + By + Cz + D = 0 \)) of the tangent plane II to the surface \( x^2 + y^2 - z^2 = 1 \) at the point \( P_0 = (1, 1, 1) \).

**Solution.** \( \nabla f(x, y, z) = (2x, 2y, -2z) = (2, 2, -2). \) Thus we can use the vector \( \overrightarrow{n} = (1, 1, -1) \) as the normal vector. Therefore the equation of the plane is

\[ (x, y, z) \cdot \overrightarrow{n} = (1, 1, 1) \cdot \overrightarrow{n}, \]

\[ x + y - z = 1 + 1 - 1 = 1. \]

Thus the answer is: \( x + y - z = 1 = 0. \)

4. Find the critical points of the function

\[ f(x, y) = 2x^2 - \frac{1}{3}y^3 + xy^2 - 3x \]

and determine which of them are local minima, maxima, saddle points.

**Solution.** The function is differentiable on the whole plane, there are no boundary points, hence the only critical points are the stationary points.

\[ \nabla f(x, y) = (4x + y^2 - 3, -y^2 + 2xy). \] The stationary points satisfy:

\[ \nabla f(x, y) = (0, 0); \quad 4x + y^2 - 3 = 0, -y^2 + 2xy = 0. \]

From the 2-nd equation we get: either \( y = 0 \) or \( y = 2x. \) If \( y = 0 \) then \( x = 3/4 \) and hence \( P_0 = (3/4, 0) \) is the first stationary point. If \( y = 2x \) then \( 2y + y^2 - 3 = 0 \) which has two solutions: \( y_1 = 1, y_2 = -3. \) If \( y = y_1 = 1 \) then \( x = x_1 = 1/2. \) If \( y = y_2 = -3 \)
then $x = x_2 = -3/2$. Thus we get three critical points: $P_0 = (3/4, 0)$, $P_1 = (1/2, 1)$, $P_2 = (-3/2, -3)$. The determinant of the 2-nd derivatives is

$$D(P) = \begin{vmatrix} 4 & 2y \\ 2y & -2y + 2x \end{vmatrix}.$$ 

If $P = P_0$ we get:

$$D(3/4, 0) = \begin{vmatrix} 4 & 0 \\ 0 & 3/2 \end{vmatrix} = 6 > 0$$

Since $f_{xx}(P_0) = 4 > 0$, the point $P_0$ is the point of local minimum.

If $P = P_1$ we get:

$$D(1/2, 1) = \begin{vmatrix} 4 & 2 \\ 2 & -2 + 1 \end{vmatrix} = -4 - 4 = -8 < 0$$

hence $P_1$ is a saddle point.

If $P = P_2$ we get:

$$D(-3/2, -3) = \begin{vmatrix} 4 & -6 \\ -6 & 6 - 3 \end{vmatrix} = 12 - 36 = -24 < 0$$

hence $P_2$ is a saddle point.

5. Use Lagrange’s method to find the minimum point(s) of the function $f(x, y) = (x - 1)^2 + (y - 1)^2$ subject to the constrain $(x + 1)(y + 1) = 0$. You can assume that the minimum exists.

**Solution.** Let $g(x, y) = (x + 1)(y + 1)$, then $\nabla g(x, y) = (y + 1, x + 1)$, $\nabla f(x, y) = 2(x - 1, y - 1)$. Thus we get:

$$(x - 1, y - 1) = \lambda(y + 1, x + 1), \quad (x + 1)(y + 1) = 0.$$ 

If $\lambda \neq 0$ then $x = 1, y = 1$, which contradicts the equation $(x + 1)(y + 1) = 0$. Hence $\lambda \neq 0$. Then $(x - 1)(x + 1) = (y - 1)(y + 1)$, $x^2 = y^2$, $y = \pm x$. Substituting this to the equation $(x + 1)(y + 1) = 0$ we get: $x = y, (x + 1)^2 = 0, x = -1 = y$ or $y = -x, (x + 1)(1 - x) = 0, x^2 = 1, x = \pm 1, y = - \pm 1$.

Therefore we got three points where the minimum can occur: $P_0 = (-1, -1), P_1 = (1, -1), P_2 = (-1, 1)$.

Now we compare the values of $f$ at the points where minimum can occur. $f(P_0) = (-1-1)^2 + (-1-1)^2 = 8, f(P_1) = (1-1)^2 + (-1-1)^2 = 4, f(P_2) = (-1-1)^2 + (1-1)^2 = 4$. Since $4 < 8$ we conclude that the points $P_1 = (1, -1), P_2 = (-1, 1)$ are the points of minimum for the function $f$ on the curve $(x + 1)(y + 1) = 0$.

Note that the minimum exists since the function $f$ is square of the distance function from the point $(1, 1)$.