
April 15, 2001

1. Problems # 4, 12 from Section 16.8. Each problem is worth 20 points.

**Problem 4.** Compute the volume of the solid under the surface \( z = xy \) above the \( xy \)-plane within the cylinder \( x^2 + y^2 = 2x \). Use the cylindrical coordinates.

\[
\begin{align*}
\int_{x=0}^{x=2} \int_{y=0}^{\sqrt{2x-x^2}} \int_{0}^{xy} dz dy dx.
\end{align*}
\]

Figure 1:

**Solution.** \( xy \) is positive for \( x > 0, y > 0 \) and \( x < 0, y < 0 \). Hence the projection of the solid to the \( xy \)-plane is contained in the 1-st and 3-rd coordinate quadrants. The equation \( x^2 + y^2 = 2x \) is equivalent to \( (x - 1)^2 + y^2 = 1 \), which is the circle with the center \((1, 0)\) and radius 1. Thus the cylinder \( x^2 + y^2 = 2x \) in the 3-space is the circular cylinder over the circle \( (x - 1)^2 + y^2 = 1 \). The interior of this cylinder is given by the inequality \( (x - 1)^2 + y^2 \leq 1 \). Thus the projection \( R \) of the solid to the \( xy \)-plane is the upper half of the disk \( (x - 1)^2 + y^2 \leq 1 \) (the one, contained in the 1-st quadrant). The volume is given (in the cartesian coordinates) by the iterated integral
In cylindrical coordinates we will have the following description of the half-disk $R$:

$$0 \leq \theta \leq \pi/2, \ r^2 \leq 2r \cos(\theta),$$

Equivalently:

$$0 \leq \theta \leq \pi/2, \ r \leq 2 \cos(\theta).$$

The whole solid $S$ in the cylindrical coordinates is:

$$0 \leq z \leq xy = r^2 \cos(\theta) \sin(\theta) = \frac{r^2}{2} \sin(2\theta),$$

$$0 \leq \theta \leq \pi/2, \ r \leq 2 \cos(\theta).$$

Thus the volume equals

$$\int_0^{\pi/2} \int_0^{2 \cos(\theta)} \int_0^{r^2 \cos(\theta) \sin(\theta)} r \, dz \, dr \, d\theta =$$

$$\int_0^{\pi/2} \int_0^{2 \cos(\theta)} r^3 \cos(\theta) \sin(\theta) \, dr \, d\theta =$$

$$\int_0^{\pi/2} \frac{(2 \cos(\theta))^4}{4} \cos(\theta) \sin(\theta) \, d\theta =$$

$$\int_0^{\pi/2} 4 \cos^5(\theta) \sin(\theta) \, d\theta = \frac{2}{3} [- \cos^6(\theta)]_0^{\pi/2} = 2/3.$$  

**Problem 12.** Compute the volume of the solid within the sphere $x^2 + y^2 + z^2 = 16$ outside of the cone $z = x^2 + y^2$ and above the $xy$-plane. Use the spherical coordinates.

**Solution.** The volume is the different of the two volumes: $V = V_1 - V_2$ where $V_1$ is the volume of the upper half ($z \geq 0$) of the ball $x^2 + y^2 + z^2 \leq 16$, $V_2$ is the volume of the conical sector $z \geq x^2 + y^2$ within the ball $x^2 + y^2 + z^2 \leq 16$. Note that the angle between the cone $z = x^2 + y^2$ and the $z$-axis is $\pi/4$ (this cone is the surface of revolution obtained by revolving the line $z = x$ along the $z$-axis).

In the spherical coordinates the first volume is

$$V_1 = \int_{\rho=0}^{\rho=4} \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi/2} \rho^2 \sin(\phi) \, d\phi \, d\theta \, d\rho =$$

$$\int_0^4 \int_0^{2\pi} \rho^2 \, d\theta \, d\rho = 2\pi \, 4^3 / 3 = \frac{128\pi}{3}.$$ 

In the spherical coordinates the second volume equals

$$V_2 = \int_{\rho=0}^{\rho=4} \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi/4} \rho^2 \sin(\phi) \, d\phi \, d\theta \, d\rho =$$

$$\frac{2\pi}{3} [\rho^3]^{1/4}_{\rho=0} - \cos(\phi)]^{\pi/4}_{\rho=0} = \frac{2\pi}{3} \cdot 64(1 - \sqrt{22}) = \frac{64\pi(2 - \sqrt{2})}{3}. $$

Thus

$$V = V_1 - V_2 = \frac{128\pi}{3} - \frac{64\pi(2 - \sqrt{2})}{3} = \frac{64\sqrt{2}\pi}{3}. $$
2. Problems # 14, 20 from Section 17.1.  

**Problem 14.** (5 points) Compute $curl$ and $div$ for the vector field $F(x, y, z) = (x^2, y^2, z^2)$.

**Solution.** $curl(F) = 0$. $div(F) = 2x + 2y + 2z$.

**Problem 20.** (20 points) Show that

(a) $div(curl F) = 0$.

(b) $curl(\text{grad} f) = 0$.

(c) $div(f F) = f div(F) + (\text{grad} f) \cdot F$.

(d) $curl(f F) = f curl(F) + (\text{grad} f) \times F$. (This part we have done in the class.)

Here $F = (F_1, F_2, F_3)$ is a vector field in $\mathbb{R}^3$ with continuous 2-nd partial derivatives.

**Solution.** (a) The formal reason for this identity is $div(curl F) = \nabla \cdot (\nabla \times F) = 0$ since $\nabla \times F$ is “orthogonal to both $\nabla$ and $F$” (the latter is true for the ordinary vectors, but $\nabla$ is not!). Here is the actual computation:

$$div(curl F) =$$

$$\frac{\partial}{\partial x}(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}) =$$

$$[\frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial y \partial z}] + [\frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x}] + [\frac{\partial^2 F_1}{\partial z \partial y} - \frac{\partial^2 F_1}{\partial z \partial y}] = 0.$$  

(b) The formal comutational reason for this identity is $curl(\text{grad}) = \nabla \times \nabla = 0$. 

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Figure 2:
Here is the actual computation:

\[
\text{grad}(f) = (F_1, F_2, F_3) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).
\]

\[
\text{curl}(\text{grad}) = \left( \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right).
\]

\[
(f_{xy} - f_{yx}, f_{xz} - f_{zx}, f_{yz} - f_{zy}) = (0, 0, 0).
\]

(c) Let’s compare the lefthand side with the righthand side:

\[
\text{div}(f \vec{F}) = (f F_1)_x + (f F_2)_y + (f F_3)_z = (f_x F_1 + f_y F_2 + f_z F_3) + f \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right).
\]

\[
(\text{grad}f) \cdot \vec{F} + f \text{div}(\vec{F}) = (f_x F_1 + f_y F_2 + f_z F_3) + f \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right).
\]

Hence LHS=RHS and we are done.

3. Problems # 6, 10 from Section 17.2.

Problem 6. (10 points) Compute the line integral

\[
\int_C (x^2 + y^2 + z^2)ds
\]

where \(C\) is the curve \(x = 4 \cos(t), y = 4 \sin(t), z = 3t, 0 \leq t \leq 2\pi\).

Solution.

\[
\int_C (x^2 + y^2 + z^2)ds = \int_0^{2\pi} (16 \cos^2(t) + 16 \sin^2(t) + 9t^2) \sqrt{16 \sin^2(t) + 16 \cos^2(t) + 9} dt = \int_0^{2\pi} 5(16 + 9t^2) dt = 5[16t + 3t^3]_0^{2\pi} = 5[32\pi + 24\pi^3]
\]

Problem 10. (10 points) Compute the line integral

\[
\int_C y^3 dx + x^3 dy
\]

where \(C\) is the curve \(x = 2t, y = t^2 - 3, z = 3t, -2 \leq t \leq 1\).

Solution.

\[
\int_C y^3 dx + x^3 dy = \int_{-2}^{1} [2(t^2 - 3)^3 + 8t^3(2t)] dt = 2 \int_{-2}^{1} [t^6 - t^4 + 27t^2 - 27] dt = 2[t^7/7 - t^5/5 + 9t^3 - 27t]_{-2}^{1} = 2[1/7 - 1/5 + 9 - 27] - 2[-128/7 + 32/5 - 72 + 54]
\]